

CLASS NOTES	
Class: 12	Topic: MATRICES
Subject: APPLIED MATHEMATICS	

Top Definitions

1. Matrix is an ordered rectangular array of numbers (real or complex) or functions or names or any type of data. The numbers or functions are called the elements or the entries of the matrix.

2. The horizontal lines of elements are said to constitute, rows of the matrix and the vertical lines of elements are said to constitute, columns of the matrix.

3. A matrix is said to be a column matrix if it has only one column.

$A = [a_{ij}]_{m \times 1}$ is a column matrix of order $m \times 1$

4. A matrix is said to be a row matrix if it has only one row.

$B = [b_{ij}]_{1 \times n}$ is a row matrix of order $1 \times n$.

5. A matrix in which the number of rows is equal to the number of columns is said to be a square matrix. A matrix of order " $m \times n$ " is said to be a square matrix if $m = n$ and is known as a square matrix of order ' n '.

$A = [a_{ij}]_{m \times n}$ is a square matrix of order m .

6. If $A = [a_{ij}]$ is a square matrix of order n , then elements $a_{11}, a_{22}, \dots, a_{nn}$ are said to constitute the diagonal, of the matrix A

7. A square matrix $B = [b_{ij}]_{m \times m}$ is said to be a diagonal matrix if all its non diagonal elements are zero, that is a matrix $B = [b_{ij}]_{m \times m}$ is said to be a diagonal matrix if $b_{ij} = 0$, when $i \neq j$.

8. A diagonal matrix is said to be a scalar matrix if its diagonal elements are equal, that is, a square matrix $B = [b_{ij}]_{n \times n}$ is said to be a scalar matrix if

$$b_{ij} = 0, \quad \text{when } i \neq j$$

$$b_{ij} = k, \text{ when } i = j, \text{ for some constant } k.$$

9. A square matrix in which elements in the diagonal are all 1 and rest are all zero is called an identity matrix. A square matrix

$$A = [a_{ij}]_{n \times n} \text{ is an identity matrix, if } a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

10. A matrix is said to be zero matrix or null matrix if all its elements are zero.

11. Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if

(i) They are of the same order

(ii) Each elements of A is equal to the corresponding element of B, that is $a_{ij} = b_{ij}$ for all i and j.

12. If $A = [a_{ij}]$ be an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A. Transpose of the matrix A is denoted by A' or (A^T) .

13. For any square matrix A with real number entries, $A + A'$ is a symmetric matrix and $A - A'$ is a skew symmetric matrix.

14. If $A = [a_{ij}]_{n \times n}$ is an $n \times n$ matrix such that $A^T = A$, then A is called symmetric matrix. In a symmetric matrix, $a_{ij} = a_{ji}$ for all i and j

15. If $A = [a_{ij}]_{n \times n}$ is an $n \times n$ matrix such that $A^T = -A$, then A is called skew

symmetric matrix. In a skew symmetric matrix, $a_{ij} = -a_{ji}$
 If $i=j$, then $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

Example:

If $\begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix} = \begin{bmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{bmatrix}$ write the value of $a - 2b$.

Ans:

Give that $\begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix} = \begin{bmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{bmatrix}$

On equating, we get

$$a + 4 = 2a + 2, 3b = b + 2, a - 8b = -6$$

$$\Rightarrow a = 2, b = 1$$

$$\text{Now the value of } a - 2b = 2 - (2 \times 1) = 2 - 2 = 0$$

Example:

If A is a square matrix such that $A^2 = A$, then write the value of $7A - (I + A)^3$, where I is an identity matrix.

Ans:

$$7A - (I + A)^3 = 7A - \{I^3 + 3I^2A + 3IA^2 + A^3\}$$

$$= 7A - \{I + 3A + 3A + A^2A\} \quad [\because I^3 = I^2 = I, A^2 = A]$$

$$= 7A - \{I + 6A + A^2\} = 7A - \{I + 6A + A\}$$

$$= 7A - \{I + 7A\} = 7A - I - 7A = -I$$

Example:

Express the following matrices as the sum of a symmetric and a skew symmetric matrix:

$$(i) \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \quad (ii) \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \quad (iii) \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} \quad (iv) \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$$

Ans:

(i)

$$\text{Let } A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}, \text{ then } A = P + Q$$

$$\text{where, } P = \frac{1}{2}(A + A') \text{ and } Q = \frac{1}{2}(A - A')$$

$$\text{Now, } P = \frac{1}{2}(A + A') = \frac{1}{2} \left(\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 6 & 6 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix}$$

$$\therefore P' = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} = P$$

Thus $P = \frac{1}{2}(A + A')$ is a symmetric matrix.

$$\text{Now, } Q = \frac{1}{2}(A - A') = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$\therefore Q' = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -Q$$

Thus $Q = \frac{1}{2}(A - A')$ is a skew symmetric matrix.

Representing A as the sum of P and Q ,

$$P + Q = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} = A$$

(ii)

$$\text{Let } A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}, \text{ then } A = P + Q$$

where, $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$

$$\text{Now, } P = \frac{1}{2}(A + A') = \frac{1}{2}\left(\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 12 & -4 & 4 \\ -4 & 6 & -2 \\ 4 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$\therefore P' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = P$$

Thus $P = \frac{1}{2}(A + A')$ is a symmetric matrix.

$$\text{Now, } Q = \frac{1}{2}(A - A') = \frac{1}{2}\left(\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore Q' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -Q$$

Thus $Q = \frac{1}{2}(A - A')$ is a skew symmetric matrix.

Representing A as the sum of P and Q ,

$$P + Q = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = A$$

(iii)

Let $A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$, then $A = P + Q$

where, $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$

$$\text{Now, } P = \frac{1}{2}(A + A') = \frac{1}{2} \left(\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix}$$

$$\therefore P' = \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix} = P$$

Thus $P = \frac{1}{2}(A + A')$ is a symmetric matrix.

$$\text{Now, } Q = \frac{1}{2}(A - A') = \frac{1}{2} \left(\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 5/2 & 3/2 \\ -5/2 & 0 & 2 \\ -3/2 & -2 & 0 \end{bmatrix}$$

$$\therefore Q' = \begin{bmatrix} 0 & -5/2 & -3/2 \\ 5/2 & 0 & 2 \\ 3/2 & -2 & 0 \end{bmatrix} = -Q$$

Thus $Q = \frac{1}{2}(A - A')$ is a skew symmetric matrix.

Representing A as the sum of P and Q ,

$$P + Q = \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 5/2 & 3/2 \\ -5/2 & 0 & 2 \\ -3/2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} = A$$

Example:

Find x , if $\begin{bmatrix} x & -5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O?$

Ans:

Given that $\begin{bmatrix} x & -5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$

Since Matrix multiplication is associative, therefore $\begin{bmatrix} x & -5 & -1 \end{bmatrix} \begin{bmatrix} x+0+2 \\ 0+8+1 \\ 2x+0+3 \end{bmatrix} = O$

$$\Rightarrow \begin{bmatrix} x & -5 & -1 \end{bmatrix} \begin{bmatrix} x+2 \\ 9 \\ 2x+3 \end{bmatrix} = O$$

$$\Rightarrow [x(x+2) + (-5).9 + (-1)(2x+3)] = O$$

$$\Rightarrow [x^2 - 48] = O \Rightarrow x^2 - 48 = 0 \Rightarrow x^2 = 48$$

$$\Rightarrow x = \pm\sqrt{48} = \pm 4\sqrt{3}$$

Example 9 Find X and Y , if $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

Solution We have $(X + Y) + (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

or $(X + X) + (Y - Y) = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow 2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$

or $X = \frac{1}{2} \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$

Also $(X + Y) - (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

or $(X - X) + (Y + Y) = \begin{bmatrix} 5-3 & 2-6 \\ 0 & 9+1 \end{bmatrix} \Rightarrow 2Y = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}$

or $Y = \frac{1}{2} \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$

Example 18 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, then show that $A^3 - 23A - 40I = O$

Solution We have $A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$

So $A^3 = A.A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$

Now

$$\begin{aligned} A^3 - 23A - 40I &= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - 23 \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} + \begin{bmatrix} -23 & -46 & -69 \\ -69 & 46 & -23 \\ -92 & -46 & -23 \end{bmatrix} + \begin{bmatrix} -40 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{bmatrix} \\ &= \begin{bmatrix} 63-23-40 & 46-46+0 & 69-69+0 \\ 69-69+0 & -6+46-40 & 23-23+0 \\ 92-92+0 & 46-46+0 & 63-23-40 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O \end{aligned}$$

Example 20 If $A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, verify that

- (i) $(A')' = A$, (ii) $(A + B)' = A' + B'$,
 (iii) $(kB)' = kB'$, where k is any constant.

Solution

(i) We have

$$A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix} \Rightarrow (A')' = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} = A$$

Thus $(A')' = A$

(ii) We have

$$A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \Rightarrow A + B = \begin{bmatrix} 5 & \sqrt{3} - 1 & 4 \\ 5 & 4 & 4 \end{bmatrix}$$

Therefore $(A + B)' = \begin{bmatrix} 5 & 5 \\ \sqrt{3} - 1 & 4 \\ 4 & 4 \end{bmatrix}$

Now $A' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix}, B' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix},$

So $A' + B' = \begin{bmatrix} 5 & 5 \\ \sqrt{3} - 1 & 4 \\ 4 & 4 \end{bmatrix}$

Thus $(A + B)' = A' + B'$

(iii) We have

$$kB = k \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2k & -k & 2k \\ k & 2k & 4k \end{bmatrix}$$

Then $(kB)' = \begin{bmatrix} 2k & k \\ -k & 2k \\ 2k & 4k \end{bmatrix} = k \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix} = kB'$

Thus $(kB)' = kB'$

Example 21 If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$, $B = [1 \ 3 \ -6]$, verify that $(AB)' = B'A'$.

Solution We have

$$A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = [1 \ 3 \ -6]$$

then $AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} [1 \ 3 \ -6] = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$

Now $A' = [-2 \ 4 \ 5]$, $B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$

$$B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} [-2 \ 4 \ 5] = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix} = (AB)'$$

Clearly $(AB)' = B'A'$

THIS SHEET IS PREPARED FROM HOME