Class Notes	
Class: 12	**Topic: Integration**
Subject: Applied Mathematics	

Integral of a function

If $\frac{d(F(x))}{dx} = f(x)$, then we say that the integral or primitive or anti-derivative of f(x) w.r.t. x is F(x) and, symbolically, we write

$$\int f(x)dx = F(x).$$

In $\int f(x)dx$, x is called the variable of integration. The function f(x) is called the integrand. The symbol $\int$ stands for the integral.

If $\frac{d(F(x))}{dx} = f(x)$, then we also have $\frac{d(F(x)+C)}{dx} = f(x)$, where C is an arbitrary constant, therefore,

by definition, the integral of f(x) w.r.t. x is F(x) + C, i.e., $\int f(x)dx = F(x)+C$

The integral of f(x) w.r.t. x is not unique as c can be assigned infinitely many values. It is due to this indefinite nature of integral, we call it as indefinite integral. If C is assigned the value C_1, then $F(x) + C_1$ is a particular integral of f(x) w.r.t. x.

The process of finding integral of a function is called integration.

Hence, Integration, as understood above, is nothing but Inverse process of differentiation

Derivative Formulae	Corresponding Integral
$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n,\ n \neq -1$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ where $n \neq -1$
In particular, $\frac{d}{dx}(x) = 1$	$\int 1dx = x + C$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + C$
$\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x = a^x$	$\int a^x dx = \frac{a^x}{\log a} + C$
	$\int \frac{1}{x} dx = \log\lvert x\rvert + C$ Proof: Case 1: $x > 0$ $\frac{d(\log(\lvert x\rvert))}{dx} = \frac{d(\log x)}{dx} = \frac{1}{x}$

S. No.	Expression	Integral
1	$\int \frac{1}{\sqrt{x^2+a^2}}\,dx$	$\log\left\|x+\sqrt{x^2+a^2}\right\| + C$
2	$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$	$\log\left\|x+\sqrt{x^2-a^2}\right\| + C$
3	$\int \sqrt{x^2+a^2}\,dx$	$\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left\|x+\sqrt{x^2+a^2}\right\| + C$
4	$\int \sqrt{x^2-a^2}\,dx$	$\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left\|x+\sqrt{x^2-a^2}\right\| + C$
5	$\int \frac{1}{x^2-a^2}\,dx$	$\frac{1}{2}\log\left\|\frac{x-a}{x+a}\right\| + C$
6	$\int \frac{1}{a^2-x^2}\,dx$	$\frac{1}{2a}\log\left\|\frac{a+x}{a-x}\right\| + C$
7	$\int \sqrt{a^2-x^2}\,dx$	$\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C$

Properties of Indefinite Integral

P1. The process of differentiation and integration are inverses of each other as follows:

P1. (a) $\frac{d}{dx}\int f(x)dx = f(x)$ P1. (b) $\int \frac{d}{dx} f(x)dx = f(x) + C$

P2. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

P3. $\int k\,f(x)dx = k\int f(x)dx$

$$\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}dx = \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\cdot\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}\,dx$$

$$= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}\,dx$$

$$= \int \frac{\sqrt{x+a}}{a-b}dx - \int \frac{\sqrt{x+b}}{a-b}\,dx$$

$$= \frac{(x+a)^{3/2}}{(a-b)(\frac{3}{2})} - \frac{(x+b)^{3/2}}{(a-b)(\frac{3}{2})} + C$$

The marginal revenue of a company is given by MR = $80+20x+3x^2$, where x is the number of units sold for a period. Find the total revenue function R(x) if at x=2, R(x) = 240.

Solution: We have $\frac{d(R(x))}{dx} = 80 + 20x + 3x^2$ We find the total revenue function R(x) by integrating both sides w.r.t. x

$$\int \frac{d(R(x))}{dx} dx = \int (80 + 20x + 3x^2) dx$$

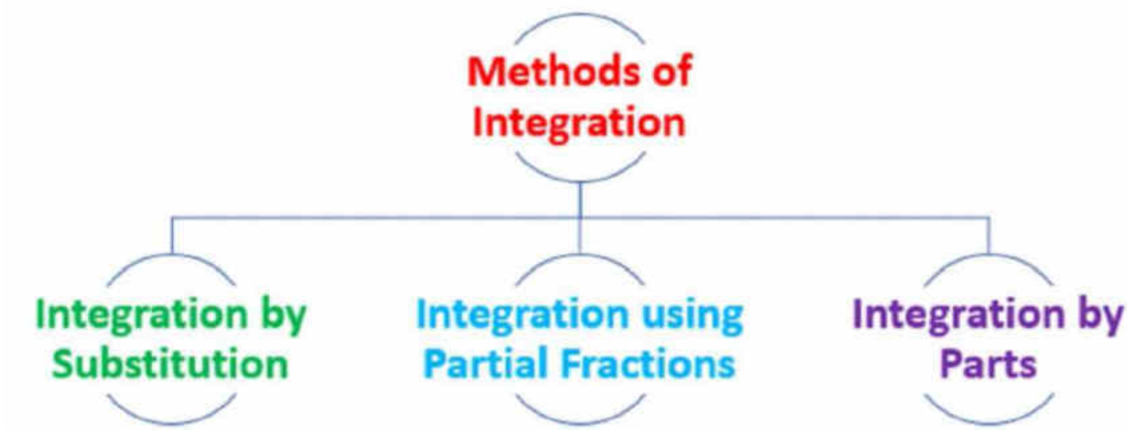
$$\Rightarrow R(x) = 80x + 10x^2 + x^3 + C.$$

The constant of integration C can be determined using the initial condition R(x=2) = 240.

Hence, $160 + 40 + 8 + C = 240 \Rightarrow C = 32$.

So, the total revenue function is given by

$R(x) = 80x + 10x^2 + x^3 + 32$.



3.2 Integration by substitution

Rule of substitution

$\int f(g(x))g'(x)dx = \int f(t)dt$, where g(x) = t

Proof:

$$\frac{d(\int f(t)dt)}{dx} = \frac{d(\int f(t)dt)}{dt} \times \frac{dt}{dx} = f(t)g'(x) = f(g(x))g'(x)$$

Integrand	Substitution
$\sqrt{f(x)}$	Put f(x) = t or t^2
logx	Put logx = t or x = e^t
fog(x) or f(g(x))	Put g(x) = t
$[f(x)]^{m/n}$	Put f(x) = t^n

Let $I = \int \frac{x}{\sqrt{x-1}}\,dx$ Let $x - 1 = t^2$ gives $dx = 2t\,dt$ and $x = t^2 + 1$

Thus, I becomes $\int \frac{t^2+1}{t}\,2t\,dt = \int 2t^2 + 2\,dt = \frac{2t^3}{3} + 2t + C$

$\therefore I = \frac{2(x-1)^{\frac{3}{2}}}{3} + 2\sqrt{x-1} + C$

Let $I = \int 3^{3^x} 3^x dx$ Put $3^x = t$ which gives $3^x \log 3 dx = dt$

I becomes $\int \frac{3^t}{\log 3} dt = \frac{3^t}{(\log 3)^2} + C = \frac{3^{3^x}}{(\log 3)^2} + C$

Alternatively, we may put $3^{3^x} = t$

Let $I = \int \frac{1}{\sqrt{5+4x+x^2}}\,dx$

Consider $5 + 4x + x^2 = 5 - 4 + 4 + 4x + x^2$ (by method of completion of squares)

$= (x + 2)^2 + 1^2$

Thus, $I = \int \frac{1}{\sqrt{(x+2)^2+1^2}}\,dx = \log|\,x + 2 + \sqrt{(x+2)^2 + 1^2}\,| + C$

$= \log|\,x + 2 + \sqrt{5 + 4x + x^2}\,| + C$

The weekly marginal cost of producing x pairs of tennis shoes is given by

$MC = 17 + \frac{200}{x+1}$, where C(x) is cost in Rupees. If the fixed costs are ₹ 2,000 per day, find the cost function.

Solution: As $MC = 17 + \frac{200}{x+1}$

$C(x) = \int MC(x)dx = \int \left[17 + \frac{200}{x+1}\right] dx = 17x + 200 \log|x+1| + C$

Given that, when $x = 0$, $C(x) = 2000$

$2000 = 17(0) + 200 \log 1 + C$ which gives $C = 2000$

Hence, $C(x) = 17x + 200\log|x+1| + 2000$

3.3 Integration by Partial Fractions

We know that a rational function is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}$, where P(x) and Q(x) are polynomials in x and $Q(x) \neq 0$.

S.No.	Type of Rational Function	Corresponding Partial Fractions Decomposition
1	$\frac{px+q}{(x+a)(x+b)}$	$\frac{A}{x+a}+\frac{B}{x+b}$
2	$\frac{px^2+qx+c}{(x+a)(x+b)(x+c)}$	$\frac{A}{x+a}+\frac{B}{x+b}+\frac{C}{x+c}$
3	$\frac{px+q}{(x+a)^2}$	$\frac{A}{x+a}+\frac{B}{(x+a)^2}$
4	$\frac{px^2+qx+c}{(x+a)(x+b)^2}$	$\frac{A}{x+a}+\frac{B}{x+b}+\frac{C}{(x+b)^2}$
5	$\frac{px^2+qx+c}{(x+a)(x2+b)}$	$\frac{A}{x+a}+\frac{Bx+C}{(x^2+b)}$

Let $\frac{1}{(x-1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+3)}$

$\therefore 1 = A(x + 3) + B(x - 1) = Ax + 3A + Bx - B$

$1 = (A + B)\ x + 3A - B$

Comparing coefficients of x and constant terms on both sides we get

$A + B = 0$ and $3A - B = 1$

Solving we get, $A = \frac{1}{4}$ and $B = -\frac{1}{4}$

Let $I = \int \frac{1}{(x-1)(x+3)}\, dx = \frac{1}{4}\int \frac{1}{(x-1)} dx - \frac{1}{4}\int \frac{1}{(x+3)} dx$

$$= \frac{1}{4}\log|x-1| - \frac{1}{4}\log|x+3| + C$$

3.4 Integration by Parts

Now we will discuss one more method of integration, that can be used in integrating products of functions.

If u and v are any two differentiable functions of a single variable x (say). Then, by the product rule of differentiation, we have

$$\frac{d}{dx}(uv) = v\frac{d}{dx}(u) + u\frac{d}{dx}(v)$$

Integrating both sides w.r.t. x, we get

$$uv = \int v\frac{d}{dx}(u)dx + \int u\frac{d}{dx}(v)dx \Rightarrow \int u\frac{d}{dx}(v)dx = uv - \int v\frac{d}{dx}(u)dx \qquad \text{................(i)}$$

Let I = $\int xe^{2x}dx$

Assuming x as first function and e^{2x} as second function, and applying by parts, we get

I = $x \int e^{2x}dx - \int [\frac{d}{dx}(x)\int e^{2x}dx]dx + C$

$= x.\frac{e^{2x}}{2} - \int 1.\frac{e^{2x}}{2} + C$

$= x.\frac{e^{2x}}{2} - \frac{e^{2x}}{4} + C$

Let I = $\int \log x\, dx = \int 1.(\log x)dx$

Assuming logx as first function and 1 as second function, and applying by parts, we get

I = $\log x \int 1\, dx - \int[\frac{d}{dx}(\log x).\int 1\, dx]dx + C$

$= (\log x).\ x - \int \frac{1}{x}.(x)dx + C$

$= (\log x).\ x - x + C$

Integral of the type: $\int e^x[f(x) + f'(x)]dx = f(x)e^x + C$

a) Let I = $\int e^x[x^2 + 2x]dx$, Here $f(x) = x^2$ and $f'(x) = 2x$

$\therefore$ by the rule $\int e^x[f(x) + f'(x)]dx = f(x)e^x + C$ we get, $I = e^x x^2 + C$

b) Let I = $\int e^x\left[\log x + \frac{1}{x}\right]dx$, Here $f(x) = \log x$ and $f'(x) = \frac{1}{x}$

$\therefore$ by the rule $\int e^x[f(x) + f'(x)]dx = f(x)e^x + C$ we get, $I = e^x \log x + C$

c) Let I = $\int e^x\left[\frac{x-1}{x^2}\right]dx = \int e^x\left[\frac{x}{x^2} - \frac{1}{x^2}\right]dx = \int e^x\left[\frac{1}{x} + \frac{-1}{x^2}\right]dx$

$= e^x\frac{1}{x} + C$

3.5 Definite Integral

So far in this topic, we have studied about the indefinite integrals and discussed a few methods of evaluating these. In this particular section, we shall define definite integral of a function.

A definite integral is denoted by $\int_a^b f(x)dx$ where *a* is called the lower limit of the integral and *b* is called the upper limit of the integral.

Definite Integral has a fixed value.

First fundamental theorem of Integral Calculus

Theorem 1 : Let f be a continuous function on the closed interval $[a, b]$ and let A (x) be the area function. Then $A'(x) = f(x)$, for all $x \in (a, b)$.

Second fundamental theorem of Integral Calculus

Following theorem enables us to evaluate definite integrals by making use of anti-derivative.

Theorem 2 : Let f be continuous function defined on the closed interval $[a, b]$ and F(x) be an anti-derivative of f(x). Then $\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$.

Example 13

Evaluate the following definite integrals

a) $\int_0^4 |x-2|\,dx$ b) $\int_{-1}^2 |x^3 - x|\,dx$

Solution: a) Let $I = \int_0^4 |x-2|\,dx$

We know that $|x-2| = \begin{cases} (x-2), & x \geq 2 \\ -(x-2), & x < 2 \end{cases}$

$$I = \int_0^2 |x-2|\,dx + \int_2^4 |x-2|\,dx \quad \text{by } P_3$$

$$= \int_0^2 -(x-2)dx + \int_2^4 (x-2)dx$$

$$= -\frac{(x-2)^2}{2}\Big|_0^2 + \frac{(x-2)^2}{2}\Big|_2^4$$

$$= -\left[\frac{(2-2)^2}{2} - \frac{(0-2)^2}{2}\right] + \left[\frac{(4-2)^2}{2} - \frac{(2-2)^2}{2}\right]$$

$$= 2 + 2 = 4$$

b) Let $I = \int_{-1}^2 |x^3 - x|\,dx$

Let $x^3 - x = 0$ gives $x = -1, 0, 1$

Clearly, $|x^3 - x| = \begin{cases} -(x^3 - x), & x \leq -1 \\ (x^3 - x), & -1 \leq x \leq 0 \\ -(x^3 - x), & 0 \leq x \leq 1 \\ (x^3 - x), & x \geq 1 \end{cases}$

As $0, 1 \in (-1, 2)$

We may write, $I = \int_{-1}^0 |x^3 - x|\,dx + \int_0^1 |x^3 - x|\,dx + \int_1^2 |x^3 - x|\,dx$ by P_3

$$= \int_{-1}^0 (x^3 - x)dx + \int_0^1 -(x^3 - x)dx + \int_1^2 (x^3 - x)dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]\Big|_{-1}^0 - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]\Big|_0^1 + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]\Big|_1^2$$

$$= \left[\frac{0^4}{4} - \frac{0^2}{2}\right] - \left[\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right] - \left[\frac{1^4}{4} - \frac{1^2}{2}\right] + \left[\frac{0^4}{4} - \frac{0^2}{2}\right] + \left[\frac{2^4}{4} - \frac{2^2}{2}\right] - \left[\frac{1^4}{4} - \frac{1^2}{2}\right]$$

$$= \frac{1}{4} + \frac{1}{4} + 2 + \frac{1}{4} = \frac{11}{4}$$

3.7 CONSUMERS' SURPLUS AND PRODUCERS' SURPLUS

CONSUMERS' SURPLUS (CS)

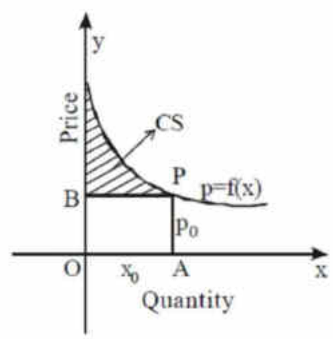
Let us first recall Demand Curve

What Is the Demand Curve?

The demand curve is a graphical representation of the relationship between the price of a good or service and the quantity demanded for a given period of time. In a typical representation, the price will appear on the left vertical axis, the quantity demanded on the horizontal axis.

Understanding the Demand Curve

We know that as per the law of demand as the price of a given commodity increases, the quantity demanded decreases, all else being equal. Thus, the demand curve will move downward from the left to the right as shown in the figure given below:



Thus, Consumers' Surplus, CS = [Total area under the demand function bounded by $x = 0$, $x = x_0$ and x-axis – Area of the rectangle OAPB]

$$\therefore CS = \int_0^{x_0} f(x)dx - p_0x_0$$

Example 17

Find the consumers' surplus for the demand function $p = 25 - x - x^2$ when $p_0 = 19$.

Solution: Given that, the demand function is $p = 25 - x - x^2$, $p_0 = 19$

$\therefore 19 = 25 - x - x^2$

$\Rightarrow x^2 + x - 6 = 0$

$\Rightarrow (x + 3)(x - 2) = 0$

$\Rightarrow x = 2$ (or) $x = -3$

$\therefore x_0 = 2$ [demand cannot be negative]

$\therefore p_0x_0 = 19 \times 2 = 38$

$$CS = \int_0^2 (25 - x - x^2)dx - 38 = 25x - \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^2 - 38 = 50 - 2 - \frac{8}{3} - 38 = \frac{22}{3}$$

Example 18

The demand function for a commodity is $p = \frac{10}{x+1}$.

Find the consumers' surplus when the prevailing market price is 5.

Solution: Given that, Demand function, $p = \frac{10}{x+1}$

$$p_0 = 5 \Rightarrow 5 = \frac{10}{x+1} \Rightarrow x = 1 \quad i.e.\ x_0 = 1$$

$$p_0 x_0 = 5$$

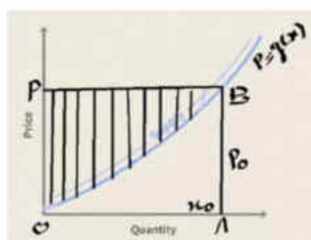
$$CS = \int_0^1 \frac{10}{x+1}dx - p_0 x_0$$

$$= 10\,[\log(x+1)]\big|_0^1 - 5.$$

$$= 10[\log 2 - \log 1] - 5 = 10 \log 2 - 5$$

However, there are producers who are willing to supply the commodity at a price lower than p_0. All such producers will gain from the fact that the prevailing market price is only p_0. This gain is called 'Producers' Surplus'.

It is represented by the area above the supply curve $p = g(x)$ and below the line $p = p_0$ as shaded in figure below.



Thus, Producers' Surplus, PS = [Area of the whole rectangle OAPB - Area under the supply curve bounded by $x = 0$, $x = x_0$ and x - axis]

i.e. $PS = p_0 x_0 - \int_0^{x_0} g(x)dx$

Example 19

The supply function for a commodity is $p = x^2 + 4x + 5$ where x denotes supply. Find the producers' surplus when the price is 10.

Solution: Given that, Supply function, $p = x^2 + 4x + 5$

For $p_0 = 10$, we have $10 = x^2 + 4x + 5 \Rightarrow x^2 + 4x - 5 = 0$

$\Rightarrow (x+5)(x-1) = 0 \Rightarrow x = -5$ or $x = 1$

Since supply cannot be negative, $x = -5$ is not possible.

$\Rightarrow x = 1$

As $p_0 = 10$ and $x_0 = 1$?$p_0 x_0 = 10$

Producers' Surplus, $PS = p_0 x_0 - \int_0^{x_0} g(x)dx = 10 - \int_0^1 (x^2 + 4x + 5)dx$

$$= 10 - [\frac{x^3}{3} + 2x^2 + 5x|_0^1\ ] = 10 - [\ \frac{1}{3} + 2 + 5] = \frac{8}{3}$$