CLASS NOTES

Class: 12	Topic: Linear Programming Problem(LPP)
Subject: Applied Mathematics	

8.1 LINEAR PROGRAMMING PROBLEM:

A Linear programming problem (LPP) consists of three important components:

(i) Decision variables

(ii) The Objective function

(iii) The Linear Constraints

1. **Decision Variable**: - The decision variables refer to the limitations or the activities that are competing with one another for sharing the available resources. These variables are usually inter-related in terms of utilization of resources and need simultaneous solution. All the decision variables are considered to be continuous, controllable and non-negative and represented as variables x, y etc.

2. **The Objective function: -** As every linear programming problem is aimed to have an objective to be measured in quantitative terms such as profit (sales) maximization, cost (time) minimization and so on. The relationship among the variables representing objective must be linear.

A linear objective is a real valued function, represented as $Z = ax + by$, where a, b are arbitrary constants, where Z is to be maximized or minimized.

3. **The Constraints**: - There are always certain limitations (constraints) on the use of resources, such as labor, space, availability of raw material or restrictions on transportation variables etc. that limit the extent to which an objective can be achieved. Such constraints are expressed as linear inequalities or equalities in terms of decision variables.

The conditions $x \geq 0$, $y \geq 0$ are called non-negative restrictions on the decision variables.

8.3 TYPES OF LINEAR PROGRAMMING PROBLEMS

The application of LPP can be found in various daily life situations.

Some of the important LP problems we shall study are:

1. Manufacturing Problem
2. Diet problem
3. Transportation Problem
4. Assignment Problem

Manufacturing Problem - These problems involve the production and sale of different products by a company. The production of the products requires optimization of labour force, machine hours, raw material, storage space, etc. Different products are produced to satisfy the aforementioned constraints and the investment available.

Diet Problem - Very often the dieticians and nutritionists are required to prepare health and diet charts. The objective of these diet charts is to include all the important kinds of nutrients that are required by the human body to stay healthy–at a reasonable cost. Thus, in the diet problems, a minimum amount of available nutrients, thereby minimizing the cost of such a diet plan.

Transportation Problem - These problems are related to the study of the efficient transportation routes i.e. how efficiently the product from different sources of production is transported to the different markets, such as the total transportation cost is minimized. Analysis of such problem is very crucial for big companies with several production plants and a widespread area to cater to. In this type of problem, constraints mean the specific supply and demand patterns and objective function means the transportation cost should be minimized.

8.4 SOLVING A LINEAR PROGRAMMING PROBLEM

In this section we are going to learn how to solve an LPP. Let us understand a few terms used while solving it.

Solution: The set of values of decision variables x_j (j = 1, 2,....,n) which satisfy the constraints of an LP problem is said to constitute solution to that LP problem.

Feasible Solution: The set of values of decision variables x_j (j = 1, 2,....,n) which satisfy all the constraints and non-negativity condition of an LP problem is said to constitute feasible solution to that LP problem.

In other way, a solution that also satisfies the non-negativity restrictions of a LPP, is called a feasible solution.

Infeasible Solution: The set of values of decision variables x_j (j = 1, 2,....,n) which do not satisfy all the constraints and non-negativity condition of an LP problem is said to constitute the infeasible solution to that LP problem.

Feasible region: Feasible region is the common region determined by all the constraints including non-negative constraints of a LPP and every point in this region is the feasible solution of the given LPP.

Optimal Feasible Solution: A feasible solution of a LPP that optimizes (maximizes or minimizes) the objective function is called the optimal solution of the LPP. At times, an LPP can have no solution or more than one optimal solution.

Theorem 1: Let R be the feasible region for a linear programming problem and let

$Z = ax + by$ be the objective function.

Theorem 2 : Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function.

(i) If R is **bounded,** then the objective function Z has both a **maximum** and a **minimum value** on R and each of these occurs at a corner point of R.

(ii) If R is **unbounded, then** maximum or minimum value of objective function may not exit. However, if it exits then it must occur at the corner point of the feasible region.

GRAPHICAL METHOD OF SOLVING LINEAR PROGRAMMING PROBLEM:

8.5.1 CORNER - POINT METHOD

In this method, the coordinates of all corner (extreme) points of the feasible region are determined and the value of the objective function at these points are computed because the mathematical theory of LP states that an optimal solution to any LP problem always lie at one of the corner points of the feasible region.

This method consists of the following steps:

(i) Formulate the given LPP in mathematical form.

(ii) Draw X-axis and Y- axis on the graph paper, the non -negativity restrictions i.e., $x \geq 0$, $y \geq 0$ imply that the values of the variables x and y can lie only in first quadrant.

(iii) Plot the inequality constraints on the graph and decide the area of feasible region according to the inequality sign of constraints.

Feasible region will always be in the first quadrant

(iv) Shade the common region of the graph that satisfies all the constraints. The common region is called the feasible region of the given LPP. Any point on or inside the feasible region is the feasible solution of the given LPP. The feasible region can be bounded (closed) or unbounded (open) as shown below:

(v) Now determine the coordinates of corner points of the feasible region

(vi) Now evaluate the objective function Z at each corner point of the feasible region. The point where the objective function attains its optimum (maximum or minimum) value is the optimal solution of the given LP problem.

Example 6

Solve the following Linear Programming Problem Graphically.

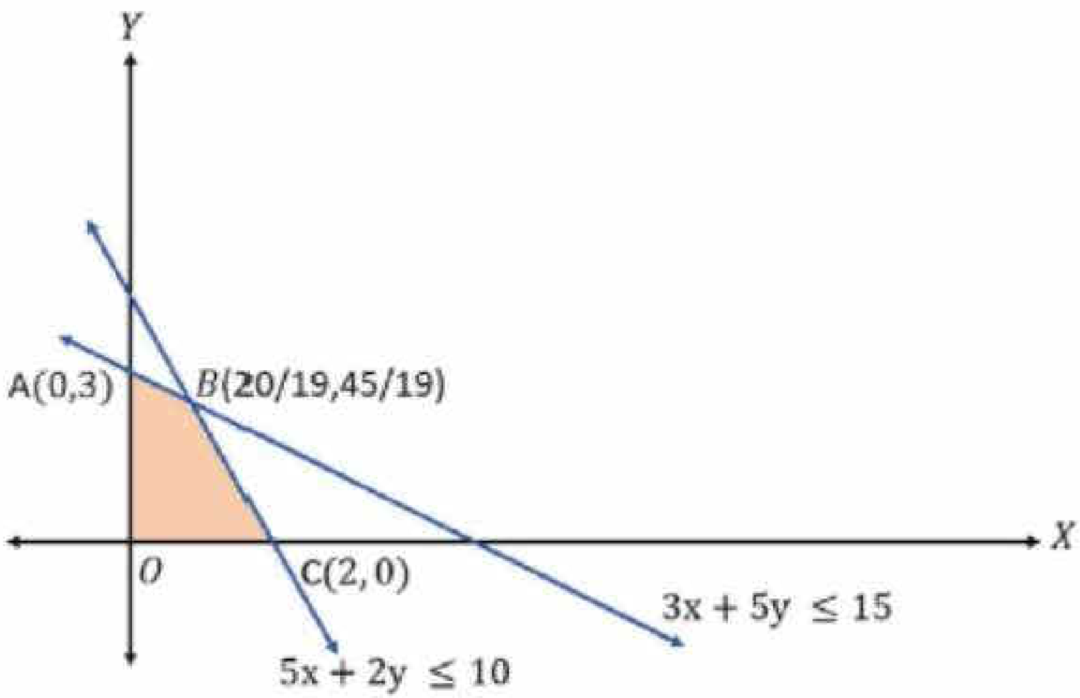
Maximize $Z = 5x + 3y$

Subject to constraints:

$3x + 5y \le 15$

$5x + 2y \le 10$

$x \ge 0$, and $y \ge 0$



Graph (1)

The coordinates of the vertices (corner point) of the shaded bounded feasible region are O (0, 0), A (0, 3), B (20/19, 45/19) and C (2, 0).

These points have been obtained by solving the equations of the corresponding intersecting lines, simultaneously. The value of the objective function as these points are given in the following table:

Corner Points	Coordinates	Objective Function $Z = 5x + 3y$
O	(0,0)	0
A	(0,3)	9
B	(20/19,45/19)	235/19
C	(2,0)	10

Clearly, Z is the maximum at P (20/19, 45/19)

Hence, $x = 20/19$, $y = 45/19$ is the optimal solution of the given LPP.

The optimal maximum value of Z is 235 /19 when $x = 20/19$ and $y = 45/19$

Example 7

Solve the following Linear Programming Problem Graphically.

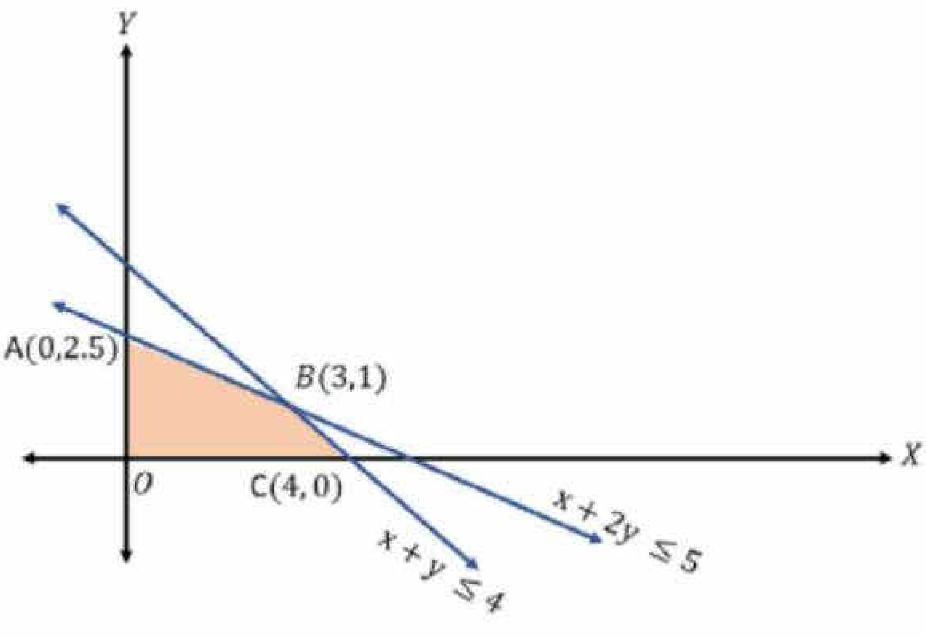
Maximize $Z = 2x + 4y$

Subject to constraints:

$x + 2y \leq 5$

$x + y \leq 4$

$x \geq 0$ and $y \geq 0$



The coordinates of the vertices (corner point) of the shaded feasible region are O (0, 0), A (0, 2.5), B (3, 1) and C (4, 0).

The value of the objective function as these corner points are given in the following table:

Corner Points	Coordinates	Objective Function $Z = 2x + 4y$
O	(0,0)	0
A	(0,2.5)	10 (Max.)
B	(3,1)	10 (Max.)
C	(4,0)	8

Clearly, Z has maximized at two corner points A (0, 2.5) and B (3, 1).

Hence, any point on the line segment joining points A and B will give the maximum value $Z = 10$ of the objective function.

The optimal maximised value of Z is 10 when $x = 0$ and $y = 2.5$ or when $x = 3$ and $y = 1$

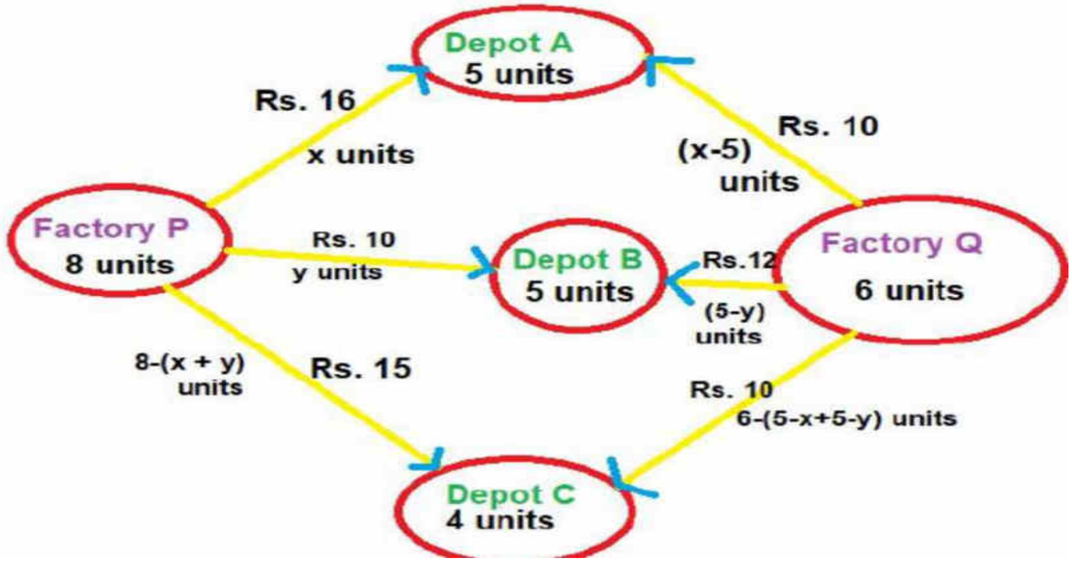
Example 4

There is a factory located at each of the two places P and Q .From these locations, a certain commodity is derived to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 5 ,5 and 4 units of the commodity while the production capacity of the factories at P and Q are 8 and 6 units respectively. The cost of transportation per unit is given below:

From/to	Costs (in Rs)		
	A	B	C
P	16	10	15
Q	10	12	10

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. Formulate above as a linear programming problem.

Solution: The above given problem can be represented in diagrammatically as follows:



Let the factory at P transports x units of commodity to depot at A and y units to depot at B.

Since the requirements are always non negative quantities. Therefore, $x \geq 0$, and $y \geq 0$

Also, the factory at P has the capacity of 8 units of the commodity.

Therefore, the left over (8-x-y) units will be transported to depot at C

Clearly, $8 - x - y \geq 0$

$\Rightarrow \quad x + y \leq 8$

Since the weekly requirement of the depot at A is 5 units of the commodity and x units are transported from the factory at P.

Therefore, the remaining quantity of $(5 - x)$ units are to be transported from the factory at Q.

Similarly, $(5 - y)$ units of the commodity will be transported from the factory at Q to the depot at B.

But the factory at Q has the capacity of 6 units only, therefore the remaining units

6-(5-x+5-y) =x+y-4 units will be transported to the depot at C.

As the requirements of the depots at A, B and C are always non negative.

$$x - 5 \geq 0, \;\; 5 - y \geq 0, \;\; \textit{and} \;\; x + y - 4 \geq 0$$

$$\Rightarrow \quad x \leq 5,$$

$$y \leq 5,$$

$$\textit{and} \quad x + y \geq 4$$

The transportation cost from the factory at P to the factory at A, B and C are respectively Rs.16x, 10y and 15(8- x -y).

Similarly, the transportation cost from the factory at Q to the depots at A, B and C are respectively Rs.10 (5-x), 12(5-y) and10(x + y - 4).

Therefore, the total transportation cost Z is given by:

$$Z = 16x+10y+15(8-x-y) +10(x-5) +12(5-y) +10(x+y-4)$$

$$= x-7y+190$$

Hence, the above LPP can be stated mathematically as follows:

$$\text{Minimize } Z = x - 7y + 190$$

Subject to the constraints:

$$x \geq 0, y \geq 0$$

$$x + y \leq 8$$

$$x + y \geq 4$$

$$x, y \leq 5$$

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