

## DETERMINANTS

### (1.) Evaluation of Determinants :-

A determinant of order two is written as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad (a_{ij} \in C \forall i, j)$$

and is equal to  $a_{11}a_{22} - a_{12}a_{21}$ .

A determinant of order three is written as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad (a_{ij} \in C \forall i, j)$$

and is equal to

$$a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{31}a_{23} - a_{32}a_{23}a_{11} - a_{12}a_{33}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{31}a_{23} - a_{32}a_{23}a_{11} - a_{12}a_{33}$$

### (2.) Minors And Co-factors :-

MINOR :- Let  $A = (a_{ij})_{n \times n}$  be a square matrix of order  $n$ . Then the minor  $M_{ij}$  of the element  $a_{ij}$  of the matrix  $A$  is the determinant of the square submatrix of order  $(n-1)$  obtained by deleting  $i$ th row and  $j$ th column of  $A$ .

COFACTOR :- Let,  $A = (a_{ij})_{n \times n}$  be a square matrix of order  $n$ . Then the co-factor of the element  $a_{ij}$  of the matrix  $A$  is denoted by  $C_{ij}$  and is equal to  $(-1)^{i+j} M_{ij}$  where  $M_{ij}$  is the minor of the element  $a_{ij}$  of the matrix  $A$ .

Note that :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\text{If } A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\text{then } A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}, \quad i = 1, 2, 3$$

$$= a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j}, \quad j = 1, 2, 3.$$

$$\text{Also, } a_{11}C_{j1} + a_{12}C_{j2} + a_{13}C_{j3} = 0, \quad i \neq j$$

$$\& a_{1i}C_{1j} + a_{2i}C_{2j} + a_{3i}C_{3j} = 0, \quad i \neq j$$

The above results remain true for determinants of every order.

### PROPERTIES OF DETERMINANTS :-

1. Reflection property :- The determinant remains unchanged if its rows are changed into columns & columns into rows.
2. All-zero property :- If all the elements of a row (column) are zero, then the determinant is zero.

### Proportionality [Repetition] property :-

If the elements of a row (column) are proportional [identical] to the elements of the some other row (column), then the determinant is zero.

### Switching property :- The interchange of any two rows (columns) of the determinant changes its sign.

### Scalar Multiple property :-

If all the elements of a row (column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.

### Sum property :-

$$\begin{vmatrix} a_1 + b_1 & c_1 & d_1 \\ a_2 + b_2 & c_2 & d_2 \\ a_3 + b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

### Property of Invariance :-

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + \alpha b_1 + \beta c_1 & b_1 & c_1 \\ a_2 + \alpha b_2 + \beta c_2 & b_2 & c_2 \\ a_3 + \alpha b_3 + \beta c_3 & b_3 & c_3 \end{vmatrix}$$

That is, a determinant remains unaltered under an operation of the form  $C_i \rightarrow C_i + \alpha C_j + \beta C_k$ , where  $j, k \neq i$  or, an operation of the form  $R_i \rightarrow \alpha R_j + \beta R_k$ , where  $j, k \neq i$ .

### Factor property :- If a determinant $\Delta$ becomes zero when we put $x = \alpha$ , then $(x - \alpha)$ is a factor of $\Delta$ .

### Triangle property :-

If all the elements of a determinant above or below the main diagonal consists of zeros, then the determinant is equal to the product of the diagonal elements. i.e.

$$\text{i.e., } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3.$$

### Property of skew-symmetric determinant :-

If  $\Delta$  is skew-symmetric determinant of odd order, then  $\Delta = 0$ .

## 11. Product of Two determinants:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} x_1 & 0 & y_1 \\ x_2 & \beta_1 & y_2 \\ x_3 & \beta_2 & y_3 \end{vmatrix} = \begin{vmatrix} a_1x_1 + b_1\beta_1 + c_1y_1 & a_1x_2 + b_1\beta_2 + c_1y_2 & a_1x_3 + b_1\beta_3 + c_1y_3 \\ a_2x_1 + b_2\beta_1 + c_2y_1 & a_2x_2 + b_2\beta_2 + c_2y_2 & a_2x_3 + b_2\beta_3 + c_2y_3 \\ a_3x_1 + b_3\beta_1 + c_3y_1 & a_3x_2 + b_3\beta_2 + c_3y_2 & a_3x_3 + b_3\beta_3 + c_3y_3 \end{vmatrix}$$

Here, we have multiplied rows by rows. we can also multiply rows by columns or columns by rows, or columns by columns.

## 12. Conjugate of a determinant :-

If  $a_i, b_i, c_i \in C (i=1, 2, 3)$ ,

and  $Z = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$  then  $\bar{Z} = \begin{vmatrix} \bar{a}_1 & \bar{b}_1 & \bar{c}_1 \\ \bar{a}_2 & \bar{b}_2 & \bar{c}_2 \\ \bar{a}_3 & \bar{b}_3 & \bar{c}_3 \end{vmatrix}$

## 12. Differentiation of a Determinant :-

If each  $a_i(x)$  is differentiable func. and  $\Delta(x) = \begin{vmatrix} a_1(x) & a_2(x) \\ a_3(x) & a_4(x) \end{vmatrix}$

then  $\Delta'(x) = \begin{vmatrix} a'_1(x) & a'_2(x) \\ a'_3(x) & a'_4(x) \end{vmatrix} + \begin{vmatrix} a_1(x) & a'_2(x) \\ a_2(x) & a'_4(x) \end{vmatrix}$

If we write  $\Delta x = [c_1, c_2]$ , where  $c_i$  denotes the  $i$ th column, then

$\Delta'(x) = [c'_1, c'_2] + [c_1, c'_2]$  where  $c'_i$  denotes the column which contains the derivative of all the functions in the  $i$ th column  $c_i$ , similarly,

$$\Delta(x) = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \text{ then } \Delta'(x) = \begin{bmatrix} r'_1 \\ r'_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Next, if  $\Delta(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21}(x) & a_{22}(x) & a_{23}(x) \\ a_{31}(x) & a_{32}(x) & a_{33}(x) \end{vmatrix}$  then

$$\Delta'(x) = \begin{vmatrix} a'_{11}(x) & a'_{12}(x) & a'_{13}(x) \\ a'_{21}(x) & a'_{22}(x) & a'_{23}(x) \\ a'_{31}(x) & a'_{32}(x) & a'_{33}(x) \end{vmatrix} + \begin{vmatrix} a_{11}(x) & a'_{12}(x) & a'_{13}(x) \\ a_{21}(x) & a'_{22}(x) & a'_{23}(x) \\ a_{31}(x) & a'_{32}(x) & a'_{33}(x) \end{vmatrix} + \begin{vmatrix} a_{11}(x) & a_{12}(x) & a'_{13}(x) \\ a_{21}(x) & a_{22}(x) & a'_{23}(x) \\ a_{31}(x) & a_{32}(x) & a'_{33}(x) \end{vmatrix}$$

$$= [c'_1, c_2, c_3] + [c_1, c'_2, c_3] + [c_1, c_2, c'_3]$$

Similarly, if

$$\Delta(x) = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \text{ then } \Delta'(x) = \begin{bmatrix} r'_1 \\ r'_2 \\ r'_3 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

Note:-

$$\text{If } \Delta(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

where,  $a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$  are constants, then

$$\Delta'(m) = \begin{vmatrix} a_{11}(m) & a_{12}(m) & a_{13}(m) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{and} \quad \int \Delta(m) dm = \begin{vmatrix} \int a_{11}(m) dm & \int a_{12}(m) dm & \int a_{13}(m) dm \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

In general, for any positive integer 'm'

$$\Delta^m(m) = \begin{vmatrix} a_{11}^m(m) & a_{12}^m(m) & a_{13}^m(m) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

#### 14. Determinant of cofactor matrix :-

$$\text{If } A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ then } \Delta_1 = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \Delta^2$$

where  $C_{ij}$  denotes the co-factor of the element  $a_{ij}$  in  $A$ .

Q.1. If  $w \neq 1$ , is a cube root of unity and

$$A = \begin{vmatrix} w+w^2 & w & 1 \\ w & w^2 & 1+w \\ 1 & w+w^2 & w^2 \end{vmatrix} = 0, \text{ then value of 'w' is}$$

- (a) 0      (b) 1      (c) -1      (d) NOT.

Soln:- Applying  $c_1 \rightarrow c_1 + c_2 + c_3$ , we get

$$A = \begin{vmatrix} w+w^2+w & w & 1 \\ w+w^2+1+w & w^2 & 1+w \\ 1+w+w^2 & w+w^2 & w^2 \end{vmatrix} = \begin{vmatrix} w & w & 1 \\ w & w^2 & 1+w \\ w & w+w^2 & w^2 \end{vmatrix} = 0.$$

So, option 'a'

Q.2. The no. of distinct real roots of :

$$A = \begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

in the interval  $-\pi/4 \leq x \leq \pi/4$ .

- (a) 0      (b) 2      (c) 1      (d) 3.

Soln:- Using  $C_j \rightarrow C_j + C_{j+1} + C_{j+2}$ , we can write

$$A = \begin{vmatrix} \sin x + 2\cos x & \cos x & \cos x \\ \sin x + 2\cos x & \sin x & \cos x \\ \sin x + 2\cos x & \cos x & \sin x \end{vmatrix} = (\sin x + 2\cos x) \begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix}$$

Applying  $R_2 \rightarrow R_2 - R_1$  &  $R_3 \rightarrow R_3 - R_1$ , we get

$$\Delta = (\sin x + 2 \cos x) \begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & \sin x - \cos x \end{vmatrix} = (\sin x + 2 \cos x)(\sin x - \cos x)^2$$

thus,  $\Delta = 0 \Rightarrow \tan x = -2$  or,  $\tan x = 1$ .

As,  $-\pi/4 \leq x \leq \pi/4$ , we get  $-1 \leq \tan x \leq 1$ .

$$\therefore \tan x = 1 \Rightarrow x = \pi/4.$$

Q.2. If  $a_1 b_1 c_1, a_2 b_2 c_2, a_3 b_3 c_3$  are 3-digit even natural numbers

and  $\Delta = \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}$ , then  $\Delta$  is

(a) divisible by 2 but not necessarily by 4.

(b) divisible by 4 but not necessarily by 8

(c) divisible by 8

(d) N.O.T.

Soln:- As  $a_1 b_1 c_1, a_2 b_2 c_2, a_3 b_3 c_3$  are even natural numbers, each of  $c_1, c_2, c_3$  is divisible by 2. Let,  $c_i = 2k_i$  for  $i=1, 2, 3$ . thus

$$\Delta = 2 \begin{vmatrix} k_1 & a_1 & b_1 \\ k_2 & a_2 & b_2 \\ k_3 & a_3 & b_3 \end{vmatrix} = 2m$$

where 'm' is some integer. thus,  $\Delta$  is divisible by 2. that  $\Delta$  may not be divisible by 4 can be seen by taking the three numbers as 112, 122 & 124. Note that

$$\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 2 & 1 & 2 \\ 4 & 1 & 3 \end{vmatrix} = 2$$

which is divisible by 2 but not by 4.

Q.4. If  $\alpha, \beta, \gamma$  are the roots of  $\alpha^3 + \alpha m^2 + b = 0$  then the determinant

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} \text{ equals}$$

- (a)  $-\alpha^3$  (b)  $\alpha^3 - 2b$  (c)  $\alpha^2 - 2b$  (d)  $\alpha^3$

Soln:- we have  $\alpha + \beta + \gamma = -a$  and  $\beta\gamma + \gamma\alpha + \alpha\beta = 0$ .

using  $C_1 \rightarrow C_1 + C_2 + C_3$ , we get

$$\Delta = \begin{vmatrix} \alpha + \beta + \gamma & \beta & \gamma \\ \alpha + \beta + \gamma & \gamma & \alpha \\ \alpha + \beta + \gamma & \alpha & \beta \end{vmatrix} = -\alpha \begin{vmatrix} 1 & \beta & \gamma \\ 1 & \gamma & \alpha \\ 1 & \alpha & \beta \end{vmatrix} = -\alpha \begin{vmatrix} 1 & \beta & \gamma \\ 0 & \gamma - \beta & \alpha - \gamma \\ 0 & \alpha - \beta & \beta - \gamma \end{vmatrix}$$

using  $C_2 \rightarrow C_2 - C_1$  &  $C_3 \rightarrow C_3 - C_1$ ,

$$\begin{aligned}
 &= -a [-(\gamma-\beta)^2 - (\alpha-\gamma)(\beta-\gamma)] \\
 &= a [\alpha^2 + \beta^2 + \gamma^2 - (\beta\gamma + \gamma\alpha + \alpha\beta)] \\
 &= a [(\alpha+\beta+\gamma)^2 - 2(\beta\gamma + \gamma\alpha + \alpha\beta)] \\
 &= a [(-a)^2 - 2 \times 0] = a^3
 \end{aligned}$$

Q.5. Let 'm' be a positive integer and  $\Delta_r = \begin{vmatrix} 2r-1 & m(r) & 1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix}$

$(0 \leq r \leq m)$ . Then the value of  $\sum_{r=0}^m \Delta_r$  is given by

- (a) 0      (b)  $m^2-1$       (c)  $2^m$       (d)  $2^m \sin^2(2^m)$

Soln:- Using sum property,

$$\sum_{r=0}^m \Delta_r = \begin{vmatrix} \sum_{r=0}^m (2r-1) & \sum_{r=0}^m m(r) & \sum_{r=0}^m 1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix}$$

But,  $\sum_{r=0}^m (2r-1) = \frac{1}{2}(m+1)(2m-1-1) = (m^2-1)$ ,  $\sum_{r=0}^m m(r) = 2^m$  &  $\sum_{r=0}^m 1 = m+1$ .

Therefore,  $\sum_{r=0}^m \Delta_r = \begin{vmatrix} m^2-1 & 2^m & m+1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix} = 0$

Students must try questions given below:-

Q.6. If  $p\lambda^4 + q\lambda^2 + r\lambda^2 + s\lambda + t = \begin{vmatrix} \lambda^4 + 2\lambda & \lambda - 1 & \lambda + 3 \\ \lambda^2 + 1 & 2 - \lambda & \lambda - 3 \\ \lambda^2 - 2 & \lambda + 4 & 2\lambda \end{vmatrix}$  then 'p' is equal to

- (a) -5      (b) -4      (c) -3      (d) -2.

Q.7. Let  $f(x) = \begin{vmatrix} \sin x & \cos x & \sin x + \tan x \cos x \\ \cos^2 x & \cos x & \cos x \\ 1 & \cos x & \cos x \end{vmatrix}$  then value of  $\int_{\pi/4}^{\pi/2} f(x) dx$  is

- (a) 0      (b)  $\pi/48$       (c)  $-\frac{\pi}{2} - \frac{1}{15\sqrt{2}}$       (d) NOT.

Q.8. For fixed positive integer  $n$ , let  $D = \begin{vmatrix} (n+1)! & (n+1)! & (n+2)! / n(n+1) \\ (n+1)! & (n+2)! & (n+3)! / (n+2)(n+3) \\ (n+2)! & (n+3)! & (n+4)! / (n+3)(n+4) \end{vmatrix}$

Then  $\frac{D}{(n+1)! (n+1)! (n+2)!}$  is equal to ...

- (a) -8      (b) -16      (c) -32      (d) -64.