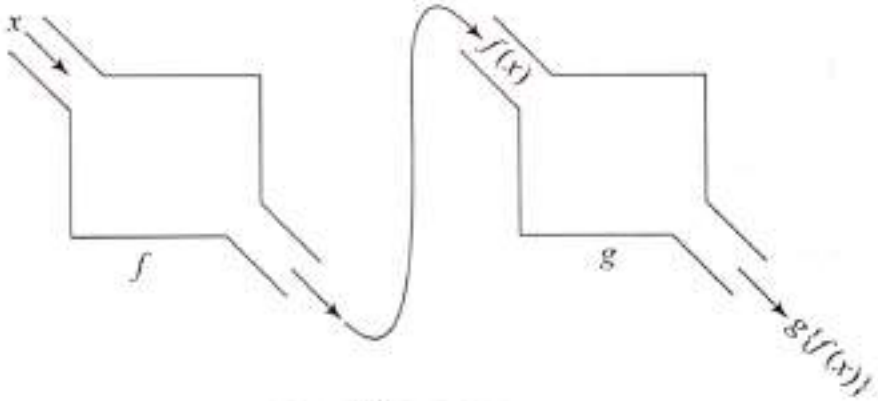
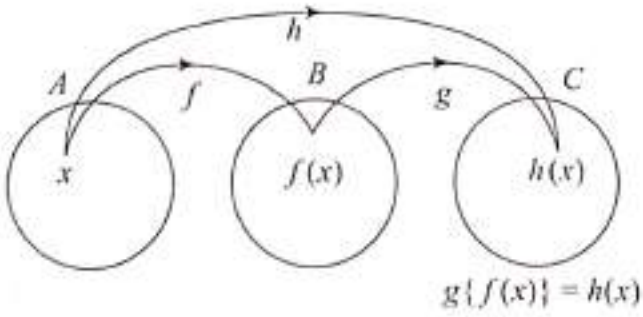
2.9 Composite Functions

Let us consider two maps $f:A \to B$ and $g:B \to C$. We can define a new map $h:A \to C$ as follows: For any $x \in A$ we can define $h(x) = g\{f(x)\}$. The function 'h' so defined is called the composite of f & g and written as $h = gof(x)$. Thus $gof(x) = g\{f(x)\}$. It should be followed clearly that to each $x \in A$ the function 'gof' associates the g-value of the f-value of x. The composite function is also called function of function.





Ex.1. Let $f(x) = \sin x$ and $g(x) = \sqrt{|1-x|}$. Find $gof(x)$, $fog(x)$, $gog(x)$ & $fof(x)$.

Sol. $gof(x) = g\{f(x)\} = g(\sin x) = \sqrt{|1-\sin x|}$.

$fog(x) = f\{g(x)\} = f\left\{\sqrt{|1-x|}\right\} = \sin\sqrt{|1-x|}$.

$gog(x) = g\{g(x)\} = g\left\{\sqrt{|1-x|}\right\} = \sqrt{\left|1-\sqrt{|1-x|}\right|}$.

$fof(x) = f\{f(x)\} = f(\sin x) = \sin(\sin x)$.

Ex.2. If $f(x) = \dfrac{1}{1-x}$. Find $f[f\{f(x)\}]$ and draw it's graph.

Sol. $f(x) = \dfrac{1}{1-x}$. It is defined when $x \neq 1$.

Now, $f\{f(x)\} = f\left(\dfrac{1}{1-x}\right) = \dfrac{1}{\left(1-\dfrac{1}{1-x}\right)} = \dfrac{1}{\left(\dfrac{1-x-1}{1-x}\right)}$

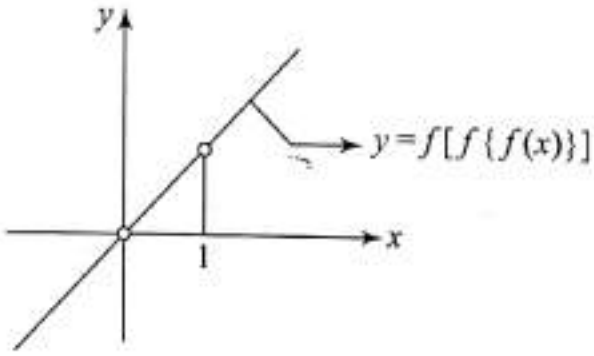
$= \dfrac{1-x}{-x}$ when $x \neq 1$; also $x \neq 0$

$= \dfrac{x-1}{x}$ when $x \neq 0, 1$

$\therefore f[f\{f(x)\}] = \dfrac{1}{1-\left(\dfrac{x-1}{x}\right)}$

$= \dfrac{x}{x-x+1}, \quad x \neq 0, 1$

$= x$ when $x \neq 0, 1$.



Ex.3. If $f(x) = x+1; 0 \le x \le 2$ and $g(x) = |x|; 0 \le x \le 3$. Calculate $(fog)x$, $(fof)x$, $(gof)x$ and $(gog)x$.

Sol. $(fog)x = f\{g(x)\} = f(|x|); 0 \le x \le 3$

$= |x|+1 \quad ; 0 \le |x| \le 2 \ \& \ 0 \le x \le 3$

$= |x|+1 \quad ; -2 \le x \le 2 \ \& \ 0 \le x \le 3$

$= |x|+1 \quad ; 0 \le x \le 2$

$(fof)x = f\{f(x)\}$

$= f(x+1) \quad ; 0 \le x \le 2$

$= (x+1)+1 \; ; 0 \le (x+1) \le 2 \;\&\; 0 \le x \le 2$

$= x+2 \qquad ; -1 \le x \le 1 \;\&\; 0 \le x \le 2$

$= x+2 \qquad ; 0 \le x \le 1$

Rest have been left as an exercise for students.

Ex.4. If $f(x) = \left(a - x^n\right)^{1/n}$, where $a > 0$ and n is a positive integer then show that $f\{f(x)\} = x$.

Sol. $f(x) = \left(a - x^n\right)^{1/n}$

$\therefore f\{f(x)\} = \left[a - \{f(x)\}^n\right]^{1/n}$

$= \left[a - \left\{\left(a - x^n\right)^{1/n}\right\}^n\right]^{1/n}$

$= \left[a - \left(a - x^n\right)\right]^{1/n} = \left(x^n\right)^{1/n} = x$

$\Rightarrow f\{f(x)\} = x.$

Ex.5. If $f(x) = \cos(\log x)$, then find the value of $f(x)f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$.

Sol. $f(x) = \cos(\log x)$

$\therefore \quad f(x)f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$

$= \quad \cos(\log x)\cos(\log y) - \frac{1}{2}\left[\cos\log\left(\frac{x}{y}\right) + \cos\log(xy)\right]$

$= \quad \cos(\log x)\cos(\log y) - \frac{1}{2}\left[\cos(\log x - \log y) + \cos(\log x + \log y)\right]$

Let $\log x = \alpha$ and $\log y = \beta$

$\therefore$ The given expression

$= \cos\alpha\cos\beta - \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$

$= \cos\alpha\cos\beta - \cos\alpha\cos\beta = 0$

$\Rightarrow \quad f(x)f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right] = 0.$

Ex.6. If $\quad f(x) = -1 + |x-1| \; ; -1 \le x \le 3$

$g(x) = 2 - |x+1| \quad ; -2 \le x \le 2$

then calculate $(fog)x$ & $(gof)x$. Draw their graphs.

Sol. $(fog)x = f\{g(x)\}$

$= f\{2 - |x+1|\} \; ; -2 \le x \le 2$

$= f(u)$ where $u = 2 - |x+1|$

$= -1 + |u-1|$; where $-1 \le u \le 3$ & $-2 \le x \le 2$

$= -1 + \left|2 - |x+1| - 1\right|$; where $-1 \le 2 - |x+1| \le 3$ & $-2 \le x \le 2$ ...(i)

Now let us solve $-1 \le 2 - |x+1| \le 3$

$\Rightarrow \quad -3 \le -|x+1| \le 1 \Rightarrow \quad 3 \ge |x+1| \ge -1$

$\Rightarrow \quad |x+1| \ge -1 \quad$ and $\quad |x+1| \le 3$

$\Rightarrow \quad x \in R \quad$ and $\quad -3 \le x+1 \le 3$

$\Rightarrow \quad x \in R \quad$ and $\quad -4 \le x \le 2$

$\Rightarrow \quad -4 \le x \le 2$

$\therefore$ The region in which $(fog)x$ is defined is the intersection of $-4 \le x \le 2$ & $-2 \le x \le 2$ ...(ii)

$\therefore \quad (fog)x = -1 + \left|1 - |x+1|\right|; -2 \le x \le 2$

Calculation of $(gof)x$ is left for the students.

Ex.7. If $f(x) = 1 + x; 0 \le x \le 2$

$= 3 - x; 2 < x \le 3$. Determine $g(x) = f\{f(x)\}$.

Sol. $g(x) = f\{f(x)\} = \begin{cases} f(1+x); 0 \le x \le 2 & .....(i) \\ f(3-x); 2 < x \le 3 & .....(ii) \end{cases}$

Now to find $f(1+x)$ We assume $1 + x = u$.

$f(1+x) = f(u)$

$= \begin{cases} 1+u; 0 \le u \le 2 \ \& \ 0 \le x \le 2 \\ 3-u; 2 < u \le 3 \ \& \ 0 \le x \le 2 \end{cases}$

$= \begin{cases} 1+(1+x); 0 \le (1+x) \le 2 \ \& \ 0 \le x \le 2 \\ 3-(1+x); 2 < (1+x) \le 3 \ \& \ 0 \le x \le 2 \end{cases}$

$= \begin{cases} 2+x; -1 \le x \le 1 \ \& \ 0 \le x \le 2 \\ 2-x; 1 < x \le 2 \ \& \ 0 \le x \le 2 \end{cases}$

$= \begin{cases} 2+x; 0 \le x \le 1 \\ 2-x; 1 < x \le 2 \end{cases}$ ...(A)

From $f\{f(x)\} = f(3-x)$ we have

$f(3-x) = \begin{cases} 1+(3-x); 0 \le 3-x \le 2 \ \& \ 2 < x \le 3 \\ 3-(3-x); 2 < 3-x \le 3 \ \& \ 2 < x \le 3 \end{cases}$

$= \begin{cases} 4-x; 1 \le x \le 3 \ \& \ 2 < x \le 3 \\ x \quad ; 0 \le x < 1 \ \& \ 2 < x \le 3 \end{cases}$ $\quad = \begin{cases} 4-x; 2 < x \le 3 \\ x \ ; \ x = \phi \end{cases}$ ...(B)

$g(x) = f\{f(x)\} = \begin{cases} 2+x; 0 \le x \le 1 \\ 2-x; 1 < x \le 2 \\ 4-x; 2 < x \le 3 \end{cases}$

Ex.8. Determine all functions f satisfying the functional relation

$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)} \quad \forall x \in R - \{0,1\}$.

Sol. Given that $f(x)+f\left(\frac{1}{1-x}\right)=\frac{2(1-2x)}{x(1-x)}=2\left(\frac{1}{x}-\frac{1}{1-x}\right)$ ...(1)

Replacing x by $\frac{1}{1-x}$ we get $f\left(\frac{1}{1-x}\right)+f\left(\frac{1}{1-\left(\frac{1}{1-x}\right)}\right)=\frac{2}{\left(\frac{1}{1-x}\right)}-\frac{2}{1-\left(\frac{1}{1-x}\right)}$

$$\Rightarrow \quad f\left(\frac{1}{1-x}\right)+f\left(\frac{x-1}{x}\right)=2(1-x)-2\frac{x-1}{x}=-2x+\frac{2}{x} \quad ...(2)$$

Again replacing x by $\frac{1}{1-x}$, we get $f\left(\frac{1}{1-\frac{1}{1-x}}\right)+f\left(\frac{\frac{1}{1-x}-1}{\frac{1}{1-x}}\right)=-\frac{2}{1-x}+\frac{2}{\frac{1}{1-x}}$

$$\Rightarrow \quad f\left(\frac{x-1}{x}\right)+f(x)=-\frac{2}{1-x}+2(1-x)=-2x+2-\frac{2}{1-x}$$

$$=-2x+\frac{2x}{x-1} \quad ...(3)$$

Operating (1) + (3) – (2) we get $2f(x)=\frac{2x}{x-1}-\frac{2}{1-x} \Rightarrow \quad f(x)=\frac{x+1}{x-1}$

2.10 Mapping of Function

Let A and B be two non empty sets and let 'f' denote a rule, which associates every element of set A to one and only one element of set B then this rule or correspondence is called a function or mapping from the set A to the set B. This is written as $f: A \to B$ and read as 'f' maps from 'A to B'. This correspondence is denoted by $y=f(x)$.

Following are the terms frequently used in mapping.

(i) Domain of 'f'–The set A is called the domain of the function 'f'.

(ii) Co-domain of 'f'– The set B is called the co-domain of the function 'f'.

(iii) Range of 'f'– The set $\{f(x) / x \in A, f(x) \in B\}$ is called the range of the function. Clearly, range is the subset of co-domain.

(iv) y is called the image of 'f' under 'f' and x is called the pre-image of y.

Note: From the definition of function it follows that there may exist some element in set B, which may not have any corresponding element in set A. But there should not be any x left (element of A) for which there is no element in set B.

Function as a set of ordered pairs – A function is a set of ordered pairs, no two of which have the same first component for different 2nd components.

Way of representing a function

Let $A = \{1, 2, 3\}$ and $B = \{1, 4, 9,\}$

(a) As tabular form

Elements of A	1	2	3
Elements of B	1	4	9

(b) As arrow diagram from the diagram we get

$f(1) = 1; f(2) = 4; f(3) = 9$

(c) As a set of ordered pairs

$f = \{(1, 1), (2, 4), (3, 9)\}$

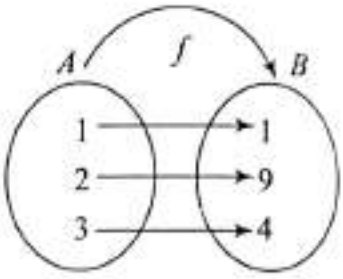
(d) As a formula or an equation $f: A \to B; f(x) = x^2$

In this case also it is clear that

$f(1) = 1^2 = 1; f(2) = 2^2 = 4; f(3) = 3^2 = 9$

The above is represented as –

$f = \{x, f(x) / x \in A \;\; f(x) = x^2\}$

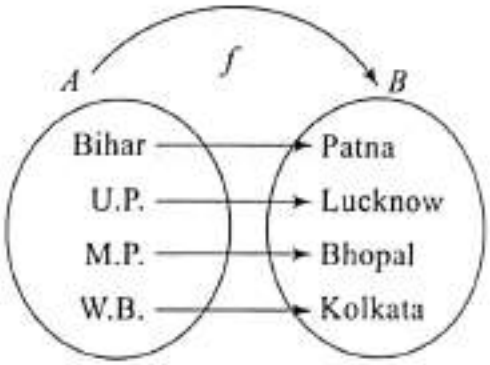


(e) **Verbal description:** The diagram shown on right, is an example of a function as for each $x \in A$ there is an unique $y \in B$. In this case we have f(state) = capital of corresponding state.

e.g. f(Bihar) = Patna.

It can be represented as

$f = \{x, y / x \in A, y$ is capital of $x\}$



Remarks: Functions are represented by (a), (b), (c) & (d) only when the set A has finite number of elements. If the number of elements of A is infinite, it will be represented by formula form or equation form and not by (a), (b), (c) & (d).

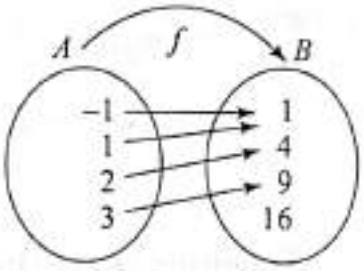
Domain, Co-domain and range of function

Let us consider the example as given in diagram on right.

Domain of 'f' = $\{-1, 1, 2, 3\}$

Co-domain of 'f' = $\{1, 4, 9, 16\}$

Range of 'f' = $\{1, 4, 9\}$



2.10.1 Definitions

(i) Into, Onto Functions or mapping

Into function: If the function $f: A \to B$ is such that there is at least one element in set B which is not the 'f' image of any element of A, then f is a mapping from A into B and symbollically expressed as $f : A \xrightarrow{\text{into}} B$

In this case range of 'f' is a proper subset of co-domain of 'f'.

Onto Function(Surjective): If the function $f: A \to B$ is such that each element of B is the f image of at least one element in A then we say that f is onto function. It is symbollically expressed as $f : A \xrightarrow{\text{onto}} B$.

In this case range of $f = B$ i.e., $f(A) = B$. It is also called surjective mapping. To show that 'f' is onto, start with any $y \in B$ and try to find $x \in A$ such that $f(x) = y$. Or, show that the range of f = co-domain.

One-One, Many-One Mapping

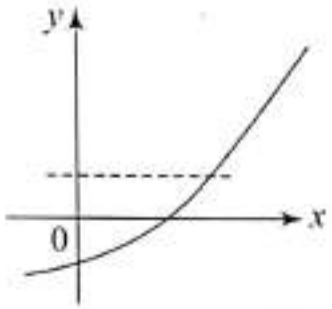
One-One or injective mapping– A function $f: A \to B$ is said to be one-one (injective) if different elements of set A have different 'f' images in set B. Thus no two elements of set A can have the same 'f' image. In other words, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Thus to prove a function one-one, start with $f(x_1) = f(x_2)$ and show $x_1 = x_2$.

Note: A function is one-one if and only if no line parallel to the x-axis meets the graph of the function at more than one point. Hence, all strictly increasing and strictly decreasing functions are one-one function.

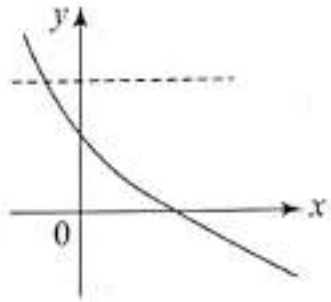
Obviously, a function may be proved one-one if we are able to prove it strictly monotonic.

Many one mapping: A function $f: A \rightarrow B$ is called many-one if at least one element in the set B is the 'f' image of more than one elements of A.

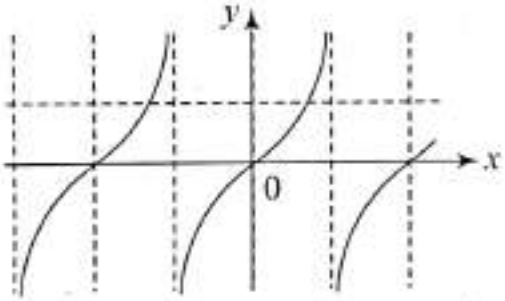
Even functions are examples of many-one because these are symmetrical about y-axis and a line parallel to x-axis cuts the graph at more than one point. Periodic functions are also many one as they have same output at x_0, $x_0 + T, x_0 + 2T$ etc. (T being fundamental period). For more clarity about one-one or many-one, see the graphs below–



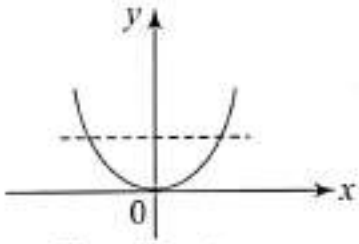
Strictly increasing function (one-one)



Strictly decreasing function (one-one)



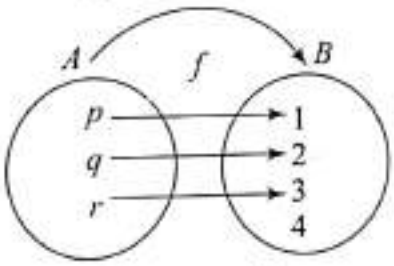
Periodic function (many-one)



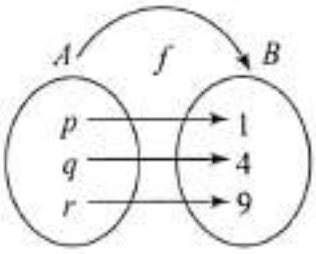
Even function (many-one)

Types of mapping– Let $f: A \rightarrow B$

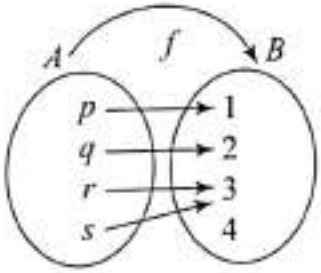
There are four types of mapping as given below–



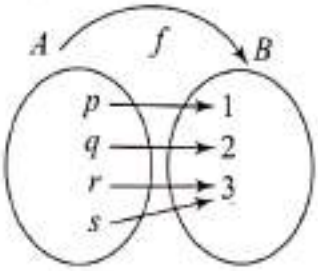
(i) one-one into



(ii) one-one onto or bijective



(iii) many-one into



(iv) many-one onto

Note: A function that is both injective and surjective is called bijective.

Ex.1. Let $f: R \to R$ defined by $f(x)=3x+5$. Show that 'f' is one-one onto.

Sol. Given $f(x) = 3x + 5$ and domain $= R$.

To test 'f' is one-one .

$f'(x)=3 > 0$ i.e. $f(x)$ increases strictly in R

$\therefore$ $f(x)$ is one-one.

Alternative:

Let $f(x_1) = f(x_2)$

$\Rightarrow$ $3x_1 + 5 = 3x_2 + 5$

$\Rightarrow$ $3x_1 = 3x_2 \Rightarrow x_1 = x_2 \Rightarrow f$ is one-one.

To test whether 'f' is onto:

From the given function 'f' we find that the range of 'f' is R.

Now because, range $= R =$ co-domain .

Hence 'f' is onto.

Ex.2. Let $A = \{x / -1 \le x \le 1\} = B$. For each of the following functions from A to B, find whether it is surjective or bijective.

(a) $f(x) = |x|$

(b) $g(x) = x|x|$

(c) $h(x) = x^3$

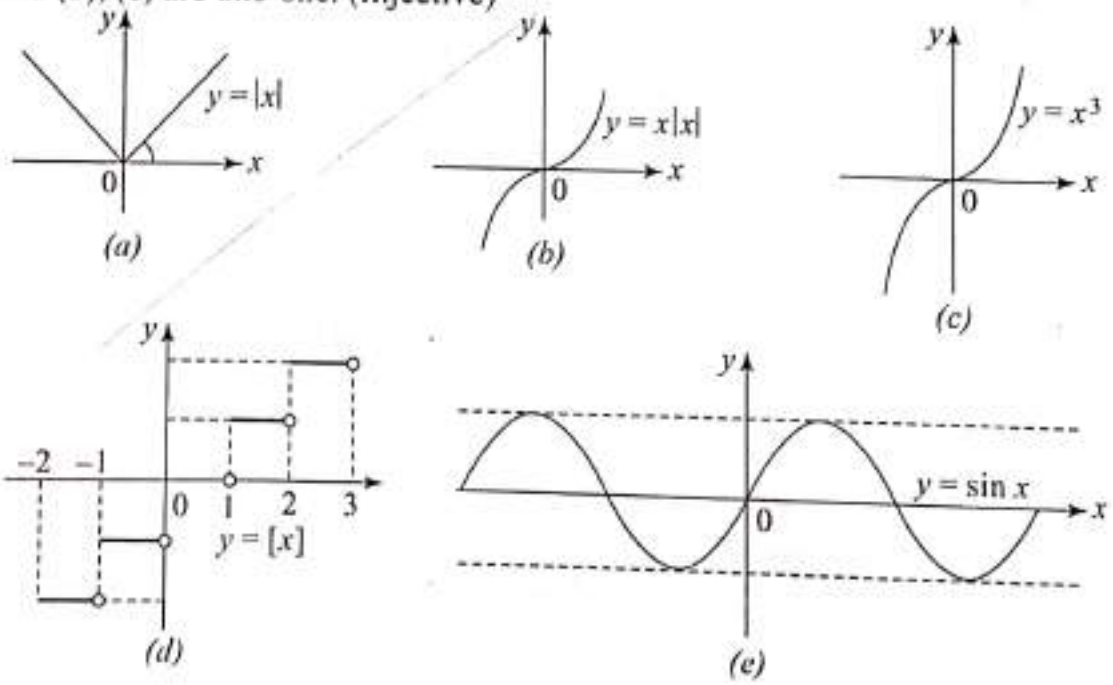
(d) $\phi(x) = [x]$

(e) $\beta(x) = \sin \pi x$.

Sol. To examine where the function 'f' is one-one.

First Method (Graphical)

Functions: (a), (d), (e), are many-one because a line parallel to x-axis cuts them at more than one point and (b), (c) are one-one. (Injective)



Second Method:

Start from $f(x_1) = f(x_2)$ and show whether $x_1 = x_2$ only or not. For instance, if we take (a) $f(x) = |x|$

Let $x_1, x_2 \in A$ & Let $f(x_1) = f(x_2)$ $\Rightarrow |x_1| = |x_2| \Rightarrow |x_1|^2 = |x_2|^2$

$\Rightarrow \quad x_1^2 = x_2^2 \quad \Rightarrow \quad x_1 = \pm x_2$ (Many-one)

If we take

(c) $h(x) = x^3$

$h(x_1) = h(x_2) \quad \Rightarrow \quad x_1^3 = x_2^3$

$\Rightarrow \quad (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$

$\Rightarrow \quad x_1 = x_2$ for real x. (Injective)

$\phi(x) = [x]$ is also many-one.

Because $\phi(x_1) = \phi(x_2) = \phi(x_3) = 0$,

if x_1, x_2 & x_3 are in $[0,1[$.

e.g., $\phi(0.1) = [0.1] = 0$

$\phi(0.2) = [0.2] = 0$ etc.

To examine whether the functions are onto.

Range of

$f(x) \quad = |x|$ is $[0,1] \neq$ co-domain of 'f'

$\quad = [-1,1]$ hence, 'f' is not onto.

Range of $g(x) = x|x|$; $h(x) = x^3$ & $\beta(x) = \sin \pi x$ is $[-1,1]$ = co-domain of g, h & β respectively i.e. $[-1,1]$

Hence, g, h & β are onto.

Range of $\phi(x) = [x] = \{-1, 0, 1\} \neq$ co-domain $[-1,1]$.

Therefore $\phi(x)$ is not onto.

Ex.3. Let $f: R \to R$ defined by $f(x) = \dfrac{x^2}{1+x^2}$ prove that 'f' is neither injective nor surjective.

Sol. To examine whether 'f' is one-one.

Because 'f' is even function in its domain, hence it is many-one.

You can also prove the same by the excution of

$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$.

To examine whether 'f' is onto.

Let us find the range of 'f'

Let $\dfrac{x^2}{1+x^2} = y \Rightarrow x^2 = y + yx^2$

$x^2(1-y) = y \Rightarrow x = \pm\sqrt{\dfrac{y}{1-y}}$

Now x is defined if $\dfrac{y}{1-y} \geq 0 \Rightarrow 0 \leq y < 1$ i.e.$[0,1[$

Because range of 'f' is $[0,1[\neq R$, co-domain of 'f'.

Hence, 'f' is not onto.

i.e. 'f' is neither one-one (injective) nor onto (surjective).

Ex.4. Let $A = R - \{3\}$, $B = R - \{1\}$ and let $f: A \to B$ defined by $f(x) = \dfrac{x-2}{x-3}$. Is 'f' bijective ? Give reasons.

Sol. Let $x_1, x_2 \in A$ and and let $f(x_1) = f(x_2)$

$$\Rightarrow \quad \frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3}$$

$$\Rightarrow \quad x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 3x_2 - 2x_1 + 6$$

$$\Rightarrow \quad x_1 = x_2 \Rightarrow f \text{ is one-one}$$

To, prove that 'f' is onto, first let us find the range of 'f'.

Let $y = f(x) = \dfrac{x-2}{x-3} \quad \Rightarrow \quad xy - 3 = x - 2$

$$\Rightarrow \quad x(y-1) = 3y - 2 \quad \Rightarrow \quad x = \frac{3y-2}{y-1}$$

x is defined if $y \neq 1$

i.e. range of 'f' is $R-\{1\}$ which is also the domain of 'f'.

Also, for no value of y, x can be 3 i.e. if we put

$$3 = x = \frac{3y-2}{y-1}$$

$$\Rightarrow \quad 3y - 3 = 3y - 2$$

$\Rightarrow \quad -3 = -2$ (not possible) Hence, 'f' is onto.

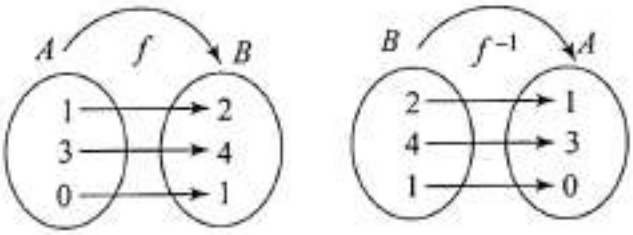
2.11 Inverse Mapping (Inverse function)

Let us consider a one-one function with domain A and range B. Let $y \in B$. This member $y \in B$ arises from one and only one member $x \in A$ such that $f(x) = y$, as the function is one-one.

Thus, we can define a new function say 'g' such that $g(y) = x \Leftrightarrow f(x) = y$.

We can also notate g by f^{-1}

i.e. $x = f^{-1}(y) \Leftrightarrow y = f(x)$. In the form of usual notations (i.e. y as a function of x), we sometimes represent inverse of $y = f(x)$ as $y = f^{-1}(x)$. In this case, domain of f^{-1} = range of 'f' and range of f^{-1} = domain of 'f'.

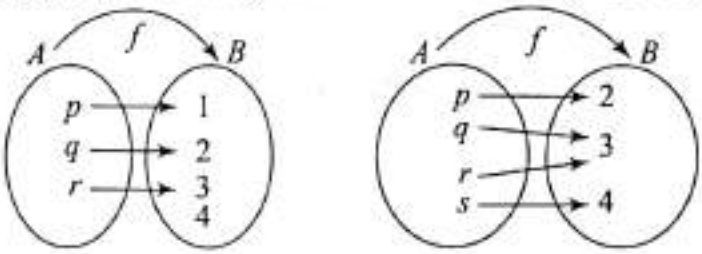


Domain of 'f'$=\{1, 3, 0\}$ = Range of f^{-1}

and Range of 'f'$=\{2, 4, 1\}$ = Domain of f^{-1}

Only one-one Onto functions are invertible

For proof, let us consider two functions namely one-one into and many-one onto as shown below :



In one-one into, we see that $f(p)=1; f(q)=2; f(r)=3$; & $f^{-1}(1)=p$; $f^{-1}(2)=q$; $f^{-1}(3)=r$; but $f^{-1}(4)$ does not exist. Hence, f is not invertible i.e. f^{-1} does not exist for into map.

In many one onto function, we see that $f(p)=2; f(q)=3; f(r)=3; f(s)=4$ and $f^{-1}(2)=p$; $f^{-1}(3)=q$ & r i.e. for one input '3' we have more than one outputs q & r, which is against the property of a function.

Hence, it is clear that only those functions are invertible which are neither into nor many one i.e. which are *one-one onto*. All one-one onto functions are strictly monotonic in nature, hence sufficient condition for the existence of invertiblity of an onto function $y=f(x)$ is that it must be strictly monotonic. If a function increases or decreases then its inverse also increases or decreases accordingly.

Illustration:

(i) $y=x^3$ with domain R is invertible and the inverse is $x=y^{1/3}$ or in the form $y=f^{-1}(x)$ it is $y=x^{1/3}$.

(ii) $y=x^2$ is not invertible in R. But it is invertible in $[0,\infty)$ where the inverse is $x=\sqrt{y}$ or in usual notations $y=\sqrt{x}$. It is also invertible in $(-\infty, 0]$ having inverse as $x=-\sqrt{y}$ or, in usual notations $y=-\sqrt{x}$

Note: We can find the inverse of many one function also but only when their domain is restricted such that in the restricted domain they behave like one-one.

2.11.2 Graph of the Inverse of an Invertible Function

Now let us consider a function $y=f(x)$ defined on the set x having a range y. If for each $y \in Y$ there exists a single value of x such that $f(x)=y$ then this correspondence defines a certain function $x=g(y)$ called inverse with respect to the given function $y=f(x)$.

Let (h, k) be a point on the graph of the function f, then (k, h) is the corresponding point on the graph of the inverse of f i.e, g.

The line segment joining the points (h, k) & (k, h) is bisected at right angles by the line

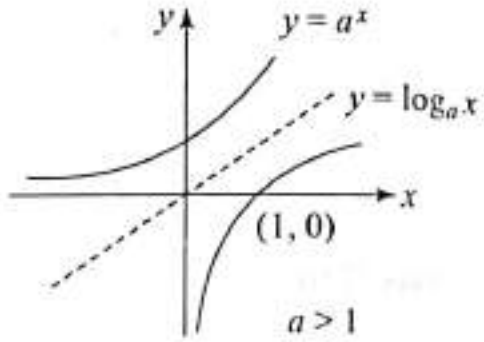
$y=x$ so that the two points play object-image role in the line $y=x$ as plane mirror.

It follows that the graph of $y=f(x)$ and its inverse written in the form $y=g(x)$ {rather than $x=g(y)$}or $y=f^{-1}(x)$ are symmetric about the line $y=x$.

The curves $y=f(x)$ and $y=f^{-1}(x)$ if intersect, they do so on the line $y=x$ generally. Hence, the solutions of $f(x)=f^{-1}(x)$ are also the solutions of $f(x)=x$.

Find the inverse of the function $y = \log_a\left(x+\sqrt{x^2+1}\right), (a>0, a\neq 1)$ (assuming onto).

The function $y = \log_a\left(x+\sqrt{x^2+1}\right)$, is defined for all x.



since $\sqrt{x^2+1} > |x|$

Now $y = \log_e\left(x+\sqrt{x^2+1}\right)\times\log_a e$

$$\therefore \quad y' = \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{2x}{2\sqrt{x^2+1}}\right).\log_a e \quad = \frac{1}{\sqrt{x^2+1}}.\log_a e$$

$\sqrt{x^2+1}$ is positive always so 'y' has sign as $\log_a e$.

If $a>1$; $\log_a e > 0$ & $y' > 0$ i.e. y is strictly increasing

If $0<a<1$; $\log_a e < 0$ & $y' < 0$ i.e. y is strictly decreasing.

Hence, the given function is invertible.

Now $y = \log_a\left(x+\sqrt{x^2+1}\right)$

$$\Rightarrow \quad a^y = x+\sqrt{x^2+1} \ \& \ a^{-y} = \sqrt{x^2+1}-x \quad \Rightarrow \quad x = \frac{1}{2}\left(a^y - a^{-y}\right)$$

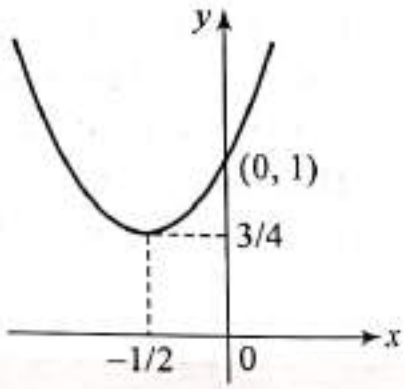
The inverse in the form $y = f^{-1}(x)$ is $y = \frac{1}{2}\left(a^x - a^{-x}\right)$

Is $f(x) = x^2+x+1$ invertible ? If not in which region is it invertible? Give brief reasons (assume $f(x)$ onto).

Since $f(x) = x^2+x+1$ is a many one function, it is not invertible in R. But the inverse can be obtained by restricting the domain.

The function y or $f(x) = x^2+x+1$ increases strictly in $[-1/2, \infty)$ and decreases strictly in $(-\infty, -1/2]$.

Hence, the inverse can be obtained when domain is restricted to either $[-1/2, \infty)$ only or $(-\infty, -1/2]$ only.



Now for inverse we have $x^2+x+1 = y$

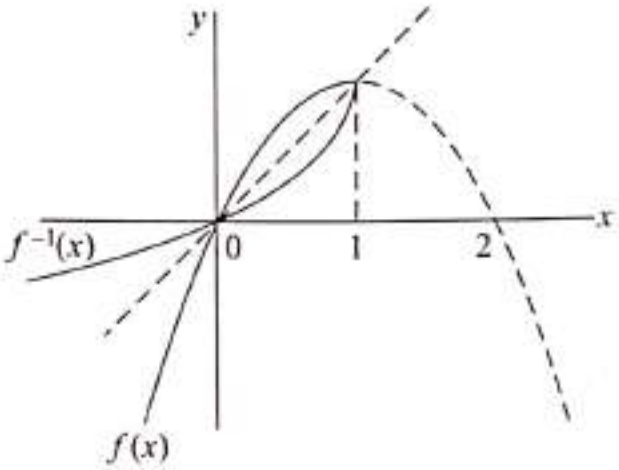
i.e. $\quad x^2+x+1-y = 0$ i.e. $x = \dfrac{-1\pm\sqrt{4y-3}}{2}$

The inverse for $x\in[-1/2, \infty)$ is $x = \dfrac{-1+\sqrt{4y-3}}{2}$ and inverse in the form

$y = f^{-1}(x)$ for this region is $y = \dfrac{-1+\sqrt{4x-3}}{2}$

Let $f(x) = 2x - x^2$; $x \le 1$. Find the roots of the equation $f(x) = f^{-1}(x)$.

Graphical::



From the graph it is clear that $f(x) = f^{-1}(x)$ at $x = 0,1$.

Analytical: We know that the curves $y = f(x)$ and $y = f^{-1}(x)$ intersect each other on the line $y = x$.

Thus solutions of $f(x) = f^{-1}(x)$ are same as that of the solutions of $f(x) = x$.

$\therefore \quad 2x - x^2 = x$

$\Rightarrow \quad x = 0, 1.$

Find the real roots of the equation $x^2 + 2ax + \frac{1}{16} = -a + \sqrt{a^2 + x - \frac{1}{16}}$ $\left(0 < a < \frac{1}{4}\right)$

Let $y = -a + \sqrt{a^2 + x - \frac{1}{16}} = x^2 + 2ax + \frac{1}{16}$

$\Rightarrow \quad (a + y)^2 = a^2 + x - \frac{1}{16}$

$\Rightarrow \quad a^2 + 2ay + y^2 = a^2 + x - \frac{1}{16}$

$\Rightarrow \quad x = y^2 + 2ay + \frac{1}{16}$

Thus $y = x^2 + 2ax + \frac{1}{16}$ and $x = y^2 + 2ay + \frac{1}{16}$ are inverse of each other.

Hence, the curves $y = x^2 + 2ax + \frac{1}{16}$ and $y = -a + \sqrt{a^2 + x - \frac{1}{16}}$ can intersect each other only on the line $y = x$. Thus the roots of the above equation are the roots of $x^2 + 2ax + \frac{1}{16} = x$

$\Rightarrow \quad x^2 + (2a - 1)x + \frac{1}{16} = 0 \qquad \Rightarrow \qquad x = \dfrac{(1 - 2a) \pm \sqrt{(1 - 2a)^2 - \frac{1}{4}}}{2}$

We see that for $0 < a < \frac{1}{4}$ both the above roots are real and hence are our solutions.