

MATRICES

1.1 THE ALGEBRA OF MATRICES :-

A matrix is a rectangular array of numbers. The numbers may be real or complex. It may be represented as -

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

or, as $A = [a_{ij}]_{m \times n}$

A matrix with m rows and n columns is called as $m \times n$ matrix and the size (or, dimension) of this matrix is said to be $m \times n$.

Note :- Two matrices are said to be equal provided they are of the same size and corresponding elements are equal. For example:

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 1 \end{bmatrix}$$

if and only if

$$a = -1, b = 2, c = 5, d = 7, e = 3 \text{ \& } f = 1$$

Some important Definitions :-

A matrix $A = [a_{ij}]_{m \times n}$ is said to be a

- (i) square matrix if $m = n$
- (ii) row matrix if $m = 1$
- (iii) column matrix if $n = 1$
- (iv) Null or, Zero matrix if $a_{ij} = 0 \forall i \& j$
- (v) diagonal matrix if $m = n \& a_{ij} = 0 \forall i \neq j$ ~~$a_{ii} = \lambda \forall i$~~
- (vi) scalar matrix if $m = n \& a_{ij} = 0 \forall i \neq j \& a_{ii} = \lambda \forall i$
- (vii) unit or, identity matrix if $m = n \& a_{ij} = 0 \forall i \neq j \& a_{ii} = 1 \forall i$
- (viii) upper (Lower) triangular matrix if $m = n \& a_{ij} = 0 \forall i > j$
($a_{ij} = 0 \forall i < j$)

Addition of Matrices :-

Two matrices $A \& B$ can be added if they are of the same size.

i.e. If $A \& B$ are two matrices of same size, then the difference betⁿ $A \& B$ is defined by

$$A - B = A + (-B)$$

Properties of Addition :-

$$A+B = B+A \quad (\text{commutative})$$

$$(A+B)+C = A+(B+C) \quad (\text{Associative law})$$

$$A+O = O+A \quad (\text{Additive property of zero})$$

$A+(-A) = O$, where 'O' is null matrix of the same size as that of A.

Note :-

$$(\alpha+\beta)A = \alpha A + \beta A$$

$$(\alpha\beta)A = \alpha(\beta A)$$

$$\alpha(A+B) = \alpha A + \alpha B,$$

where α, β are scalars & A, B are matrices of the same size.

MATRIX MULTIPLICATION :- Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ be two matrices. We say that A & B are comparable for the product AB if $n=p$, i.e. if the no. of columns of A is equal to no. of rows of B.

∴ product AB is matrix $C = [c_{ij}]_{m \times p}$ such that $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{in}b_{nj}$ for $1 \leq i \leq m, 1 \leq j \leq p$.

Properties of Multiplication :-

If $A = [a_{ij}]_{m \times n}, B = [b_{ij}]_{n \times p}, C = [c_{ij}]_{p \times q}$ then

1. $(AB)C = A(BC)$ [Associative law]

2. $AIn = ImA = A$

3. AB may not be equal to BA

4. $k(AB) = (kA)B = A(kB)$, k is a scalar.

5. If A is a square matrix, then

$$A^m A^n = A^{m+n} \quad \forall m, n \in \mathbb{N}$$

$$(A^m)^n = A^{mn} \quad \forall m, n \in \mathbb{N}$$

6. If A is an invertible matrix then

$$(A^{-1}BA)^m = A^{-1}B^m A \quad \text{and}$$

$$A^{-m} = (A^{-1})^m \quad \forall m \in \mathbb{N}$$

TRANSPOSE OF MATRIX :-

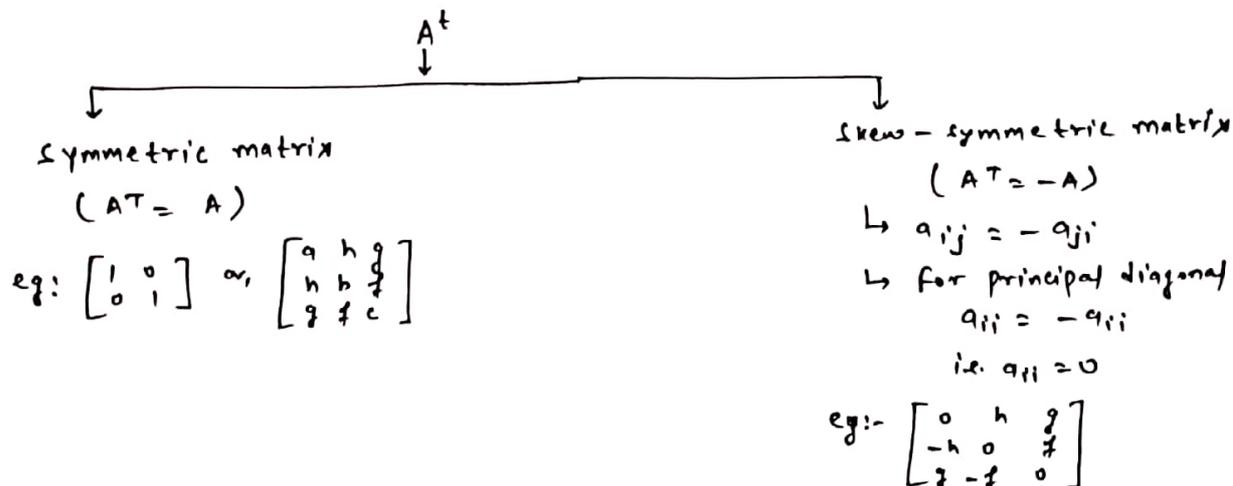
Let $A = [a_{ij}]_{m \times n}$ then Transpose of A is denoted by A' or A^t or A' where, A' or A^t or $A' = [b_{ij}]_{n \times m}$ where $b_{ij} = a_{ji}$ $\forall i \neq j$

Note :- By \bar{A} we mean $B = (b_{ij})_{m \times n}$ where $b_{ij} = \bar{a}_{ij}$ where \bar{a} denotes conjugate of a and by A^* we mean, $A^* = (\bar{A})' = (\bar{A}')$

Properties of Transpose :-

- (i) $(A+B)' = A'+B'$
- (ii) $(kA)' = kA'$ where 'k' is a scalar
- (iii) $(AB)' = B'A'$ (Reversal Law)
- (iv) If A be an invertible matrix, then $(A^{-1})' = (A')^{-1}$.

Note :-



Checking cases of Transpose for symmetric & skew symmetric :-

(1.) $A + A^T$

Let, $B = A + A^T$

$\therefore B^T = (A + A^T)^T = A^T + A = B$ (Symmetric)

(2.) $A - A^T$

Let, $B = A - A^T$

$\therefore B^T = (A - A^T)^T = A^T - A = -(A - A^T) = -B$ (Skew-symmetric)

(3.) If A is symmetric then $A^n = \dots$

solⁿ:-

Let $B = A^n \Rightarrow B^T = (A^n)^T = (A^T)^n = A^n = B$ (Symmetric)

(4.) If A is skew-symmetric then A^n is?

solⁿ:-

Let $B = A^n \Rightarrow B^T = (A^T)^n = (-A)^n \rightarrow \begin{cases} A^n & \text{if } n \text{ is even} \\ -A^n & \text{if } n \text{ is odd.} \end{cases}$

ie. symmetric if 'n' is even

& skew-symmetric if 'n' is odd.

1Q. If A & B are two symmetric matrix then check the following for symmetric or skew-symmetric.

(A) $A+B$ (B) $AB-BA$ (C) AA^T (D) $AB+BA$

Solⁿ:-

(A) let, $C = (A+B) \Rightarrow C^T = (A+B)^T = A^T + B^T = A+B$ (symmetric)

(B) let, $C = AB-BA \Rightarrow C^T = (AB-BA)^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T = BA-AB = -(AB-BA)$ (skew-sym)

(C) let, $C = AA^T \Rightarrow C^T = (AA^T)^T = AA^T$ (symmetric)

(D) let, $C = AB+BA \Rightarrow C^T = (AB+BA)^T = B^T A^T + A^T B^T = BA+AB$ (symmetric)

2Q. If $(A-B)$ is symmetric then $(A-B)^T$ is symmetric or not?

Solⁿ:- $\therefore (A-B)^T = A^T - B^T = (A-B)^T$

$\Rightarrow ((A-B)^T)^T = (A-B)$

\therefore symmetric.

Note:-

Any matrix can be represented as sum of symmetric & skew-symmetric both. (Anybody can remember this concept from functions as any function can be represented by sum of symmetric odd & even funcⁿ.)

i.e. $A = \frac{1}{2}(A+AT) + \frac{1}{2}(A-AT)$

where, $A+AT \rightarrow$ symmetric

& $A-AT \rightarrow$ skew-symmetric.

3Q. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$ then $A = P+Q$ where P is symmetric & Q is skew-symmetric. Find P & Q .

Solⁿ:- As, we know, $A = \frac{1}{2}(A+AT) + \frac{1}{2}(A-AT)$

$\therefore P = A+AT = \begin{bmatrix} 4 & 2 \\ 2 & 10 \end{bmatrix}$ & $Q = A-AT = \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$

4Q. Find minimum no. of zeros in Triangular matrix of $n \times n$ order.



$\therefore x + x + n = n^2$

$\therefore x = \frac{n^2 - n}{2}$ zeros.

S.O. Find max^m no. of distinct entries in triangular matrix of order 'n'.

Solⁿ:-

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$\therefore \frac{n^2 - n}{2} + n + 1 = \frac{n^2 + n + 2}{2}$$

⇒ Also min^m order of zero in diagonal matrix of order n is (n-1).

TRACE OF MATRIX :-

Sum of all elements of diagonal are known as Trace of matrix.

↳ It is represented by Tr(A).

↳ For n order matrix; $Tr(A) = \sum_{i=1}^n a_{ii}$

↳ possible when matrix is square matrix.

Properties :-

- (i) $Tr(A+B) = Tr(A) + Tr(B)$
- (ii) $Tr(A^n) = n Tr(A)$
- (iii) $Tr(I_n) = n$ (where I_n is unit matrix of 'n' order)
- (iv) $Tr(kA) = k Tr(A)$

S.O. If $A+2B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ -5 & 3 & 1 \end{bmatrix}$ & $2A-B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$ then find

$Tr(A) - Tr(B)$.

Solⁿ:- $Tr(A+2B) = -1 \Rightarrow Tr(A) + 2Tr(B) = -1 \dots (i)$

& $Tr(2A-B) = 3 \Rightarrow 2Tr(A) - Tr(B) = 3 \dots (ii)$

Solving (i) & (ii)

$Tr(A) = 1, Tr(B) = -1$

∴, $Tr(A) - Tr(B) = 2$

Q. If $A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$ then

$Tr(A) + Tr\left[\frac{A(BC)}{2}\right] + Tr\left[\frac{A(BC)^2}{4}\right] + \dots = ?$

Solⁿ:- $BC = I \Rightarrow A(BC) = A$
 $\Rightarrow A(BC)^2 = A$

∴ $Tr(A) + Tr\left[\frac{A(BC)}{2}\right] + Tr\left[\frac{A(BC)^2}{4}\right] + \dots = Tr(A) + Tr\left(\frac{A}{2}\right) + Tr\left(\frac{A}{4}\right) + \dots$
 $= 3 + 3/2 + 3/4 + \dots$
 $= 6$

PROPERTIES OF MATRIX MULTIPLICATION :-

- (i) If $AB = 0$ \nrightarrow $A = 0$ or, $B = 0$ But for $AB = 0 \Rightarrow A = 0$ or, $B = 0$ or, $A \& B \neq 0$
- (ii) $AB = BA$ (Commutative)
- (iii) $AB = -BA$ (Anti-commutative)
- (iv) $A(B+C) = AB+AC$
- (v) $AI = IA = A$
- (vi) $(A+B)^2 = A^2 + B^2 + AB + BA$
If A & B are commutative then $(A+B)^2 = A^2 + B^2 + 2AB$
- (vii) $AB = AC \Rightarrow B = C$
- (viii) $I^n = I$

7. Q. If $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ then prove that $A^2 = 3A - 2I$ & Hence find

A^8 in terms of A .

Solⁿ:- $A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$

$$\therefore 3A - 2I = 3 \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$$

$$\therefore, A^2 = 3A - 2I.$$

Now,

$$A^8 = ?$$

$$\text{we know, } A^2 = 3A - 2I \Rightarrow A^4 = 9A^2 + 9I^2 - 12IA \\ = 9(3A - 2I) + 9I - 12A$$

$$\therefore A^4 = 5A - 14I$$

$$\therefore, A^8 = (5A - 14I)^2 = 25A^2 + 196I^2 - 420AI = 675A - 450I \\ + 196I - 420A \\ = 255A - 254I.$$

8. Q. If A & B are two matrices such that $AB = B$ & $BA = A$ then $A^2 + B^2 =$

Solⁿ:- $A^2 + B^2 = A \cdot A + B \cdot B = A(BA) + B(AB) \\ = (AB)A + (BA)B \\ = BA + AB \\ = A + B.$

Some special multiplication:-

- (i) $A^2 = I$ (Involutory matrix)
- (ii) $A^2 = A$ (Idempotent matrix)
- (iii) $A^{k+1} = A$ (periodic matrix)
- (iv) $ATA = I$ (orthogonal matrix)
- (v) $AK = 0$ (Nilpotent matrix.)

\Rightarrow Idempotent matrix is also a periodic matrix with period 1.

Note:-

\rightarrow Diagonal matrix can be represented by $A = \text{diag}(a_1, a_2, a_3, \dots, a_n)$

\rightarrow Also, $AT = A$.

ADJOINT AND INVERSE OF A MATRIX :-

Let $A = (a_{ij})_{n \times n}$ be square matrix. The adjoint of A is defined to be the matrix $\text{adj} A = (b_{ij})_{n \times n}$ where $b_{ij} = A_{ji}$, where A_{ji} is the cofactor of (j, i) th element of A .

Properties of Adjoint :-

- (i) $A(\text{adj} A) = (\text{adj} A)A = |A|I_n$
- (ii) $\text{adj}(kA) = k^{n-1} \text{adj}(A)$
- (iii) $\text{adj}(AB) = (\text{adj} B)(\text{adj} A)$.
- (iv) $|\text{adj} A| = |A|^{n-1}$
- (v) $|\text{adj}(\text{adj} A)| = |A|^{(n-1)^2}$
- (vi) $(\text{adj} A)^T = \text{adj} A^T$
- (vii) $(\text{adj} A^n) = (\text{adj} A)^n$

Remark:- To prove these results, we need to know 'determinant', which is explained below:-

Determinant of Matrix :-

\rightarrow Represented by $\det(A)$ or $|A|$.

\rightarrow If $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ then $\det(A)$ or $|A| = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 \times 1 - 2 \times 2 = -1$

Imp:-

- (i) $|AT| = |A|$
- (ii) $|AB| = |BA|$
- (iii) $|AB| = |A||B|$
- (iv) $|kA| = k^n |A|$, n is order of sq. matrix.

Remarks:- these result doesn't cover all the properties, we will study 'Determinant' as a next chapter.

Note:- A square matrix A is said to be singular if $|A| = 0$ & non-singular if $|A| \neq 0$.

9. Q. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ then $|A|$ & $\text{adj}A = ?$

Solⁿ:- $= |A| |A|^{2-1}$ (from property)

$$= |A|^3$$

$$= (a^3)^3$$

$$= a^9$$

$$\text{Ans, } |A| = \begin{vmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = a^3$$

Remark:- If $A = \text{diag}(a_1, a_2, a_3, \dots, a_n)$, then $|A| = a_1 \times a_2 \times a_3 \times \dots \times a_n$.

10. Q. If A, B, C is matrix of order 2×2 such that $\det(A) = 2$, $\det(B) = 3$, $\det(C) = 5$ then $\det(A^2 B C^{-1}) = ?$

Solⁿ:-

$$\therefore |A^2 B C^{-1}| = |A|^2 |B| |C|^{-1} = 4 \times 3 / 5 = 12/5$$

INVERSE OF A MATRIX:-

Inverse of a square matrix 'A', is ~~said to be singular if $|A| \neq 0$~~ .

$A = (a_{ij})_{n \times n}$ is the matrix $B = (b_{ij})_{n \times n}$ such that $AB = BA = I_n$.

In fact $A^{-1} = \frac{1}{|A|} (\text{adj}A)$. So, matrix is invertible if $|A| \neq 0$.

Properties:-

(i) $AA^{-1} = A^{-1}A = I_n$

(ii) $(A^{-1})^{-1} = A$

(iii) $(kA)^{-1} = k^{-1} A^{-1}$ if $k \neq 0$

(iv) $(AB)^{-1} = B^{-1} A^{-1}$ (Reverse law)

(v) For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\text{adj}A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ & $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $ad-bc \neq 0$.

(vi) If A is a triangular matrix, then A^{-1} if it exist is a triangular matrix of the same kind.

(vii) If $A = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ then A^{-1} exists if $\lambda_i \neq 0 \forall i$ &

$$A^{-1} = \text{diag}(\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_n^{-1}) \quad \therefore \text{Also } A^m = \text{diag}(\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m) \text{ if } m \in \mathbb{N}.$$

(viii) If a sq. matrix 'A' satisfies the eqⁿ $a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = 0$, then A is invertible if $a_0 \neq 0$ and its inverse is given by $A^{-1} = \frac{1}{a_0} [a_1 I + a_2 A + \dots + a_n A^{n-1}]$

11. Q. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$, find $|\text{Adj}(\text{Adj} A)|$.

Solⁿ:- $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix} = 2(0-2) + 3(0-6) = (6-1) - 2(0-2) + 3(0-6)$
 $= 5 + 4 - 18 = -9$

$\therefore |\text{Adj}(\text{Adj} A)| = |A|^4 = 9^4$.

12. Q. If $A = \begin{bmatrix} 1 & 2 \\ 2 & -5 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ & $X = \text{matrix}$ such that $A = BX$, find X .

Solⁿ:- $A = BX \Rightarrow B^{-1}A = IX = X$

So, $X = B^{-1}A$

$\therefore B^{-1} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$

So, $X = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3/2 & -5/2 \end{bmatrix}$

13. Q. If $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then find $(f(x))^{-1} = ?$

Solⁿ:-

$\therefore (f(x))^{-1} = \frac{\text{Adj}(f(x))}{|f(x)|} = \frac{1}{1} \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(-x)$

14. Q. If A is a matrix such that $3A^3 + 2A^2 + 5A + I = 0$ then $A^{-1} = ?$

Solⁿ:- $\therefore I = -3A^3 - 2A^2 - 5A$

i.e. $A^{-1} = (-3A^2 - 2A - 5)$.

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15. Q. If M is a 2×3 matrix such that $|M| = 1$ & $M^T M = I$ then prove that $\det(M-I) = 0$.

Solⁿ:- $\det(M-I) = \det(M - M^T M) = \det(M(I - M^T)) = \det M \cdot \det(I - M^T)$

$= \det(I^T - M^T)$

$= \det(I - M)^T$

$= \det(I - M)$

$= (-1)^3 \det(M - I)$

$\therefore \det(M - I) = 0$.

16. Q. If $A = \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$ then $|A^{1002} - 5A^{1002}| = ?$

Solⁿ:- $|A^{1002} - 5A^{1002}| = |A^{1002}| |A - 5I| = 2^2 |A|^{1002} = 2^1$

16.0. If $A = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}$ then $A^{100} = ?$

Solⁿ:- $\therefore A^2 = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2(\frac{1}{2}) & 1 \end{pmatrix}$

$A^3 = \begin{pmatrix} 1 & 0 \\ 2(\frac{1}{2}) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3(\frac{1}{2}) & 1 \end{pmatrix}$

So, $A^{100} = \begin{pmatrix} 1 & 0 \\ 50 & 1 \end{pmatrix}$.

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17.0. Let X & Y be two arbitrary 2×2 , non-zero, skew-symmetric matrix & Z be an arbitrary 2×2 , non-zero symmetric matrix. Then which of the following matrices is (are) skew symmetric?

Solⁿ:- (a) $Y^2 Z^4 - Z^4 Y^2$ (b) $X^4 Y + Y^4 X$ (c) $X^4 Z^3 - Z^3 X^4$ (d) $X^{23} + Y^{23}$

Here, $X' = -X$, $Y' = -Y$, $Z' = Z$

$\therefore (Y^2 Z^4 - Z^4 Y^2)' = (Z^4)' (Y^2)' - (Y^2)' (Z^4)'$
 $= (Z^4) (Y^2)' - (Y^2)' (Z^4)$
 $= Y^2 Z^4 - Z^4 Y^2$ (skew-symm.)

Also, $X^{23} + Y^{23}$ is also skew-symmetric.

So, options c & d.

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17.0 Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ & I be the identity matrix of order 3. If

$Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\frac{q_{21} + q_{22}}{q_{21}} = ?$

Solⁿ:- Here, $P = I + \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 16 & 4 & 0 \end{bmatrix}$, Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 16 & 4 & 0 \end{bmatrix} \Rightarrow P = I + A$

Also, $A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 16 & 0 & 0 \end{bmatrix}$ & $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A^n = 0 \forall n \geq 3$

$\therefore P^{50} = (I + A)^{50} = {}^{50}C_0 I^{50} + {}^{50}C_1 I^{49} A + {}^{50}C_2 I^{48} A^2 + 0$
 $= I + 50A + 25 \times 49 A^2$

$\therefore Q = P^{50} - I = 50A + 25 \times 49 A^2$

So, $q_{21} = 50 \times 4 = 200$

& $q_{22} = 20400$

$q_{32} = 50 \times 4 = 200$

$\therefore \frac{q_{21} + q_{22}}{q_{21}} = \frac{20600}{200} = 103.$

18. Q. If $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$ & $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ then A^8 equals --

Solⁿ:- (A) $4B$ (B) $128B$ (C) $-128B$ (D) $-64B$

We have, $A = iB \Rightarrow A^2 = -B^2 = \begin{bmatrix} +2 & -2 \\ -2 & 2 \end{bmatrix} = -2B.$

$\therefore A^4 = 4B^2 = 8B$

So, $A^8 = 64B^2 = 128B$

i.e. option 'b'.

19. Q. If $P = \begin{bmatrix} \cos(\pi/6) & \sin(\pi/6) \\ -\sin(\pi/6) & \cos(\pi/6) \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP'$ then $P'Q^{2009}P =$

Solⁿ:- Here, $P' = P^{-1}$

So, $Q = PAP^{-1} = PAP^{-1}$

$\Rightarrow Q^{2009} = PA^{2009}P^{-1}$

$\therefore P'Q^{2009}P = P^{-1}(PA^{2009}P^{-1})P = A^{2009} = (I + B)^{2009}$

i.e. $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, As $B^2 = 0$, we get $B^r = 0 \forall r \geq 2$

Thus, by Binomial theorem, $A^{2009} = I + 2009B = \begin{pmatrix} 1 & 2009 \\ 0 & 1 \end{pmatrix}.$

20. Q. If $\omega \neq 1$ is cube roots of unity, then

$$A = \begin{bmatrix} 1+2\omega^{100}+\omega^{200} & \omega^2 & 1 \\ 1 & 1+\omega^{101}+2\omega^{202} & \omega \\ \omega & \omega^2 & 2+2\omega^{100}+2\omega^{200} \end{bmatrix}$$

(A) A is singular (B) A is symmetric (C) $|A| \neq 0$ (D) NOT.

Solⁿ:- Here, $A = \begin{bmatrix} \omega & \omega^2 & 1 \\ 1 & \omega & \omega \\ \omega & \omega^2 & -\omega \end{bmatrix}$ (using $1+\omega+\omega^2=0$ & $\omega^3=1$).

$\therefore |A| = 0$ (So, option 'A')

21. Q. Let, $a_k = k \binom{10}{k}$, $b_k = (10-k) \binom{10}{k}$ & $A = \begin{bmatrix} a_k & 0 \\ 0 & b_k \end{bmatrix}.$

If $A = \sum_{k=1}^9 A_k = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, find the value of $a+b$.

Solⁿ:- $a+b = \sum_{k=1}^9 (a_k + b_k) = 10 \sum_{k=1}^9 \binom{10}{k} = 10(2^{10}-2) = 10220$

22. Q. Let, $A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$ and $(A+I)^{50} - 50A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find $a+b+c+d$.

Solⁿ:- As, $A^2 = 0$, $A^k = 0 \forall k \geq 2$.

Thus, $(A+I)^{50} = I + 50A$

$\Rightarrow (A+I)^{50} - 50A = I$

$\therefore a=1, b=0, c=0, d=1.$

22. Q. If A & B are two square matrices of the same order and m is a positive integer, then

$$(A+B)^m = mC_0 A^m + mC_1 A^{m-1} B + mC_2 A^{m-2} B^2 + \dots + mC_{m-1} A B^{m-1} + mC_m B^m$$

if (a) $AB = BA$ (b) $AB + BA = 0$ (c) $A^m = 0, B^m \geq 0$ (d) NOT.

Solⁿ:- Binomial theorem is applicable if and only if $AB = BA$.

So, option 'A'.

23. Q. Let $a_k = nC_k$ for $0 \leq k \leq n$ and $A_k = \begin{bmatrix} a_{k-1} & 0 \\ 0 & a_k \end{bmatrix}$ for $1 \leq k \leq n$.

and $B = \sum_{k=1}^{n-1} A_k \cdot A_{k+1} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, then match the following:

- | | |
|-------------|--------------------------------------|
| (i) a | (a) $\frac{2^n}{n+1} \binom{2^n}{n}$ |
| (ii) $a-b$ | (b) 0 |
| (iii) $a+b$ | (c) $2^n C_{n+1}$ |
| (iv) a/b | (d) 1 |

Solⁿ:-

$$\begin{aligned} a &= a_0 a_1 + a_1 a_2 + \dots + a_{n-2} a_{n-1} = a_n a_1 + a_{n-1} a_2 + \dots + a_2 a_{n-1} \\ &= \text{no. of ways of selecting } (n+1) \text{ persons out of } n \text{ men \& } n \text{ women.} \\ &= 2^n C_{n+1} \end{aligned}$$

Similarly, $b = 2^n C_{n+1}$

$$\therefore a-b = 0 \text{ \& } a/b = 1$$

$$a+b = 2 \binom{2^n}{n+1} = \frac{2^n}{n+1} 2^n C_n$$

So, (i) \rightarrow (c), (ii) \rightarrow (b), (iii) \rightarrow (a), (iv) \rightarrow (d).

24. Q. Suppose $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies the eqⁿ: $X^2 - 4X + 3I = 0$.

Statement - I: If $a+d \neq 4$, then there are just two such matrix X .

Statement - II: There are infinite no. of matrices X , satisfying $X^2 - 4X + 3I = 0$.

$$\text{Solⁿ:- } X^2 - 4X + 3I = 0 \Rightarrow (X-I)(X-2I) = 0$$

$$\Rightarrow \begin{pmatrix} (a-1)(a-3) + bc & b(a+d-4) \\ c(a+d-4) & (d-1)(d-3) + bc \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

If $a+d \neq 4$, then $b = 0, c = 0$

$$(a-1)(a-3) = 0, (d-1)(d-3) = 0$$

$$\Rightarrow a = 1, 3, d = 1, 3$$

As $a+d \neq 4$, $a = 1, d = 1$ or $a = 3, d = 3$

Thus $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $X = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. If $a+d = 4$, we get

$$\begin{pmatrix} (a-1)(a-3) + bc & 0 \\ 0 & (d-1)(d-3) + bc \end{pmatrix} = 0. \therefore \text{there are infinite no. of matrix satisfying } X^2 - 4X + 3I = 0. \text{ (So, both are correct).}$$

25. If the matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ where a, b, c are positive real numbers such that $abc = 1$ & $ATA = I$ then find the value of $a^3 + b^3 + c^3$.

Solⁿ:- $ATA = A^2 \Rightarrow |A|^2 = 1$

So, $|A| = -(a^3 + b^3 + c^3 - 3abc) \Rightarrow (a^3 + b^3 + c^3 - 3abc)^2 = 1 \dots \textcircled{1}$

As, a, b, c are positive, $\frac{a^3 + b^3 + c^3}{3} > \sqrt[3]{a^3 b^3 c^3} = abc = 1$

$\therefore a^3 + b^3 + c^3 > 3$

\therefore from $\textcircled{1}$, $a^3 + b^3 + c^3 - 3 = 1 \Rightarrow a^3 + b^3 + c^3 = 4$

EXERCISE:-

(1.) If A is a matrix $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then $A - \frac{A^2}{2} + \frac{A^3}{9} + \dots + (-\frac{1}{2})^n A^{n+1} + \dots$
 $= \frac{7}{13} \begin{bmatrix} 1 & 9 \\ b & 1 \end{bmatrix}$.

then find $|a/b|$.

(2.) A sequence of 2×2 matrices $\{M_n\}$ is def'd as follows

$M_n = \begin{bmatrix} \frac{1}{(2n+1)!} & \frac{1}{(2n+2)!} \\ \sum_{k=0}^n \frac{(2n+2)!}{(2k+2)!} & \sum_{k=0}^n \frac{(2n+1)!}{(2k+1)!} \end{bmatrix}$, then $\lim_{n \rightarrow \infty} \det(M_n) = \lambda - e^{-1}$. Find λ .

(3.) Let $A = [a_{ij}]_{2 \times 2}$ be such that $a_{ij} = \begin{cases} 2; & \text{when } i=j \\ 0; & \text{when } i \neq j \end{cases}$, then $\left\{ \frac{\det(\text{adj}(\text{adj}A))}{5} \right\}$ equals

(4.) If $A^{-1} = \begin{bmatrix} \sin^2 \alpha & 0 & 0 \\ 0 & \sin^2 \beta & 0 \\ 0 & 0 & \sin^2 \gamma \end{bmatrix}$ & $B^{-1} = \begin{bmatrix} \cos^2 \alpha & 0 & 0 \\ 0 & \cos^2 \beta & 0 \\ 0 & 0 & \cos^2 \gamma \end{bmatrix}$ where α, β, γ

are any real nos & $C = (A^{-5} + B^{-5}) + 5A^{-1}B^{-1}(A^{-3} + B^{-3}) + 10A^{-2}B^{-2}(A^{-1} + B^{-1})$ then find $|C|$.

(5.) Let $M = [a_{ij}]_{2 \times 2}$ where $a_{ij} \in \{-1, 1\}$. Find the max^m possible value of $\det(M)$.

(6.) Let matrix $A = \begin{bmatrix} x & y & -z \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ where $x, y, z \in \mathbb{N}$. If $\det(\text{adj}(\text{adj}A)) = 28 \cdot 3^4$ then no. of such matrices A is

(7.) If A is matrix of order 3 such that $|A| = 5$ & $B = \text{adj}A$, then the value of $|\det(A^{-1}(AB)^T)|$ is equal to

(8.) Let A_n & B_n be two sq. matrices of order 3, which are def'd. as:

$A_n = [a_{ij}]$ and $B_n = [b_{ij}]$ where $a_{ij} = \frac{2i+j}{3^{2n}}$ & $b_{ij} = \frac{3i-j}{2^{2n}}$ if i, j ,

$1 \leq i, j \leq 3$. If $l = \lim_{n \rightarrow \infty} \text{Tr}(2A_1 + 3^2 A_2 + 3^3 A_3 + \dots + 2^n A_n)$ &

$m = \lim_{n \rightarrow \infty} \text{Tr}(2B_1 + 2^2 B_2 + 2^3 B_3 + \dots + 2^n B_n)$, then find the value of $\frac{l+m}{3}$.