

1. **Definition :** Rectangular array of  $mn$  numbers . Unlike determinants it has no value.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

**Abbreviated as :**  $A = [a_{ij}]$   $1 \leq i \leq m ; 1 \leq j \leq n$ ,  $i$  denotes the row and  $j$  denotes the column is called a matrix of order  $m \times n$ .

2. **Special Type Of Matrices :**

(a) **Row Matrix :**  $A = [a_{11}, a_{12}, \dots, a_{1n}]$  having one row . ( $1 \times n$ ) matrix.(or row vectors)

(b) **Column Matrix :**  $A =$  having one column. ( $m \times 1$ ) matrix (or column vectors)

(c) **Zero or Null Matrix :** ( $A = O_n$ )

An  $m \times n$  matrix all whose entries are zero .

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ is a } 3 \times 2 \text{ null matrix} \quad \& \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ is } 3 \times 3 \text{ null matrix}$$

(d) **Horizontal Matrix :** A matrix of order  $m \times n$  is a horizontal matrix if  $n > m$ .

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$$

(e) **Vertical Matrix :** A matrix of order  $m \times n$  is a vertical matrix if  $m > n$ .

(f) **Square Matrix : (Order  $n$ )** If number of row = number of column  $\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \end{bmatrix}$  a square matrix.

**Note (i)** In a square matrix the pair of elements  $a_{ij}$  &  $a_{ji}$  are called **Conjugate Elements**.

e.g.  $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

(ii) The elements  $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$  are called **Diagonal Elements** . The line along which the diagonal elements lie is called "**Principal or Leading**" diagonal.

The qty  $\sum a_{ii} =$  trace of the matrix written as , i.e.  $t_r A$

### Square Matrix

Triangular Matrix

Diagonal Matrix denote as

$d_{dia} (d_1, d_2, \dots, d_n)$  all elements

except the leading diagonal are zero

diagonal Matrix      Unit or Identity Matrix

**Note:** Min. number of zeros in a diagonal matrix of order  $n = n(n - 1)$

"It is to be noted that with square matrix there is a corresponding determinant formed by the elements of  $A$  in the same order."

3. **Equality Of Matrices :**

Let  $A = [a_{ij}]$  &  $B = [b_{ij}]$  are equal if ,

(i) both have the same order .      (ii)  $a_{ij} = b_{ij}$  for each pair of  $i$  &  $j$ .

4. **Algebra Of Matrices :**

**Addition :**  $A + B = [a_{ij} + b_{ij}]$  where  $A$  &  $B$  are of the same type. (same order)

(a) **Addition of matrices is commutative.**

i.e.  $A + B = B + A$   $A = m \times n$  ;  $B = m \times n$

(b) **Matrix addition is associative.**

$(A + B) + C = A + (B + C)$  **Note :** A, B & C are of the same type.

(c) **Additive inverse.**

If  $A + B = \mathbf{O} = B + A$   $A = m \times n$

5. **Multiplication Of A Matrix By A Scalar :**

$$\text{If } A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} ; \quad kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$$

6. **Multiplication Of Matrices : (Row by Column)**

AB exists if,  $A = m \times n$  &  $B = n \times p$   
 $2 \times 3$   $3 \times 3$

AB exists, but BA does not  $\Rightarrow AB \neq BA$

**Note :** In the product AB,  $\begin{cases} A = \text{pre factor} \\ B = \text{post factor} \end{cases}$

$$A = (a_1, a_2, \dots, a_n) \quad \& \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\begin{matrix} 1 \times n & & n \times 1 \\ AB = [a_1 b_1 + a_2 b_2 + \dots + a_n b_n] \end{matrix}$$

If  $A = [a_{ij}]$   $m \times n$  &  $B = [b_{ij}]$   $n \times p$  matrix, then  $(AB)_{ij} = \sum_{r=1}^n a_{ir} \cdot b_{rj}$

**Properties Of Matrix Multiplication :**

1. Matrix multiplication is not commutative.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} ; \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} ; AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} ; \quad BA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow AB \neq BA$  (in general)

2.  $AB = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow AB = \mathbf{O} \not\Rightarrow A = \mathbf{O} \text{ or } B = \mathbf{O}$

**Note:** If A and B are two non- zero matrices such that  $AB = \mathbf{O}$  then A and B are called the divisors of zero. Also if  $[AB] = \mathbf{O} \Rightarrow |AB| \Rightarrow |A| |B| = 0 \Rightarrow |A| = 0$  or  $|B| = 0$  but not the converse.

If A and B are two matrices such that

- (i)  $AB = BA \Rightarrow A$  and  $B$  commute each other
- (ii)  $AB = -BA \Rightarrow A$  and  $B$  anti commute each other

3. **Matrix Multiplication Is Associative :**

If A, B & C are conformable for the product AB & BC, then

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

4. **Distributivity :**

$$\left. \begin{aligned} A(B + C) &= AB + AC \\ (A + B)C &= AC + BC \end{aligned} \right\} \text{ Provided A, B \& C are conformable for respective products}$$

5. **POSITIVE INTEGRAL POWERS OF A SQUARE MATRIX :**

For a square matrix A,  $A^2 A = (AA)A = A(AA) = A^3$ .

Note that for a unit matrix I of any order,  $I^m = I$  for all  $m \in \mathbf{N}$ .

**6. MATRIX POLYNOMIAL :**

If  $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_nx^0$  then we define a matrix polynomial

$$f(A) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \dots + a_nI^n$$

where A is the given square matrix. If  $f(A)$  is the null matrix then A is called the zero or root of the polynomial  $f(x)$ .

**DEFINITIONS :**

- (a) **Idempotent Matrix :** A square matrix is idempotent provided  $A^2 = A$ .  
**Note that  $A^n = A \forall n \geq 2, n \in \mathbf{N}$ .**
- (b) **Nilpotent Matrix:** A square matrix is said to be nilpotent matrix of order m,  $m \in \mathbf{N}$ , if  $A^m = \mathbf{O}, A^{m-1} \neq \mathbf{O}$ .
- (c) **Periodic Matrix :** A square matrix is which satisfies the relation  $A^{K+1} = A$ , for some positive integer K, is a periodic matrix. The period of the matrix is the least value of K for which this holds true.  
**Note that period of an idempotent matrix is 1.**
- (d) **Involuntary Matrix :** If  $A^2 = I$ , the matrix is said to be an involuntary matrix.  
**Note that  $A = A^{-1}$  for an involuntary matrix.**

**7. The Transpose Of A Matrix : (Changing rows & columns)**

Let A be any matrix . Then,  $A = a_{ij}$  of order  $m \times n$

$$\Rightarrow A^T \text{ or } A' = [a_{ji}] \text{ for } 1 \leq i \leq n \ \& \ 1 \leq j \leq m \text{ of order } n \times m$$

**Properties of Transpose :** If  $A^T$  &  $B^T$  denote the transpose of A and B ,

- (a)  $(A \pm B)^T = A^T \pm B^T$  ; note that A & B have the same order.
- IMP. (b)**  $(AB)^T = B^T A^T$  A & B are conformable for matrix product AB.
- (c)  $(A^T)^T = A$
- (d)  $(kA)^T = kA^T$  k is a scalar .

**General :**  $(A_1, A_2, \dots, A_n)^T = A_n^T, \dots, A_2^T, A_1^T$  (reversal law for transpose)

**8. Symmetric & Skew Symmetric Matrix :**

A square matrix  $A = [a_{ij}]$  is said to be , symmetric if,

$$a_{ij} = a_{ji} \quad \forall \ i \ \& \ j \quad (\text{conjugate elements are equal}) \quad (\text{Note } A = A^T)$$

**Note:** Max. number of distinct entries in a symmetric matrix of order n is  $\frac{n(n+1)}{2}$  .

and skew symmetric if,

$$a_{ij} = -a_{ji} \quad \forall \ i \ \& \ j \ (\text{the pair of conjugate elements are additive inverse of each other})$$

**(Note  $A = -A^T$ )**

Hence If A is skew symmetric, then

$$a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0 \quad \forall \ i$$

Thus the diagonal elements of a skew symmetric matrix are all zero , but not the converse .

**Properties Of Symmetric & Skew Matrix :**

- P – 1** A is symmetric if  $A^T = A$   
A is skew symmetric if  $A^T = -A$
- P – 2**  $A + A^T$  is a symmetric matrix  
 $A - A^T$  is a skew symmetric matrix .  
Consider  $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$   
 $A + A^T$  is symmetric . Similarly we can prove that  $A - A^T$  is skew symmetric .
- P – 3** The sum of two symmetric matrix is a symmetric matrix and the sum of two skew symmetric matrix is a skew symmetric matrix .  
Let  $A^T = A$  ;  $B^T = B$  where A & B have the same order .  
 $(A + B)^T = A + B$  Similarly we can prove the other

**P – 4** If A & B are symmetric matrices then ,

(a)  $AB + BA$  is a symmetric matrix

(b)  $AB - BA$  is a skew symmetric matrix .

**P – 5** Every square matrix can be uniquely expressed as a sum of a symmetric and a skew symmetric matrix.

$$A = \underbrace{\frac{1}{2} (A + A^T)}_P + \underbrace{\frac{1}{2} (A - A^T)}_Q$$

Symmetric                  Skew Symmetric

**9. Adjoint Of A Square Matrix :**

Let  $A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$  be a square matrix and let the matrix formed by the

cofactors of  $[a_{ij}]$  in determinant  $|A|$  is  $= \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$ .

Then  $(\text{adj } A) = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$

**V. Imp. Theorem :**  $A (\text{adj } A) = (\text{adj } A) \cdot A = |A| I_n$ , If A be a square matrix of order n.

**Note :** If A and B are non singular square matrices of same order, then

(i)  $|\text{adj } A| = |A|^{n-1}$

(ii)  $\text{adj}(AB) = (\text{adj } B) (\text{adj } A)$

(iii)  $\text{adj}(KA) = K^{n-1} (\text{adj } A)$ , K is a scalar

**Inverse Of A Matrix (Reciprocal Matrix) :**

A square matrix A said to be invertible (non singular) if there exists a matrix B such that,

$$AB = I = BA$$

B is called the inverse (reciprocal) of A and is denoted by  $A^{-1}$ . Thus

$$A^{-1} = B \Leftrightarrow AB = I = BA.$$

We have ,  $A \cdot (\text{adj } A) = |A| I_n$

$$A^{-1} A (\text{adj } A) = A^{-1} I_n |A|$$

$$I_n (\text{adj } A) = A^{-1} |A| I_n \quad \therefore \quad A^{-1} = \frac{(\text{adj } A)}{|A|}$$

**Note :** The necessary and sufficient condition for a square matrix A to be invertible is that  $|A| \neq 0$ .

**Imp. Theorem :** If A & B are invertible matrices of the same order , then  $(AB)^{-1} = B^{-1} A^{-1}$ . This is reversal law for inverse.

**Note :(i)** If A be an invertible matrix , then  $A^T$  is also invertible &  $(A^T)^{-1} = (A^{-1})^T$ .

(ii) If A is invertible, (a)  $(A^{-1})^{-1} = A$  ; (b)  $(A^k)^{-1} = (A^{-1})^k = A^{-k}$ ,  $k \in \mathbb{N}$

(iii) If A is an Orthogonal Matrix.  $AA^T = I = A^T A$

(iv) A square matrix is said to be **orthogonal** if,  $A^{-1} = A^T$ .                  (v)  $|A^{-1}| = \frac{1}{|A|}$

**SYSTEM OF EQUATION & CRITERIAN FOR CONSISTENCY**

**GAUSS - JORDAN METHOD**

$$x + y + z = 6$$

$$x - y + z = 2$$

$$2x + y - z = 1$$

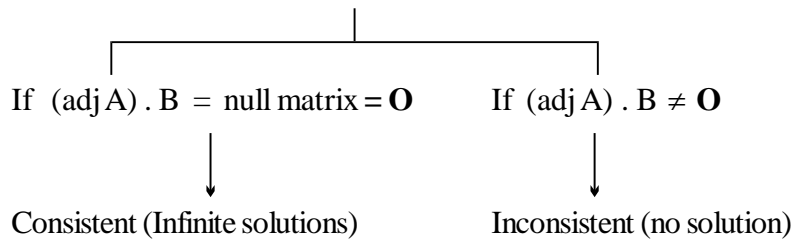
$$\text{or } \begin{pmatrix} x+y+z \\ x-y+z \\ 2x+y-z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$

$$AX = B \quad \Rightarrow \quad A^{-1} A X = A^{-1} B$$

$$X = A^{-1} B = \frac{(\text{adj. } A) \cdot B}{|A|}.$$

- Note :**
- (1) If  $|A| \neq 0$ , system is consistent having unique solution
  - (2) If  $|A| \neq 0$  &  $(\text{adj } A) \cdot B \neq \mathbf{O}$  (Null matrix), system is consistent having unique non-trivial solution.
  - (3) If  $|A| \neq 0$  &  $(\text{adj } A) \cdot B = \mathbf{O}$  (Null matrix), system is consistent having trivial solution.
  - (4) If  $|A| = 0$ , **matrix method fails**



## EXERCISE-4

Q1. Given that  $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 1 & -1 & 3 \end{bmatrix}$ ,  $C = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ ,  $D = \begin{bmatrix} 10 \\ 13 \\ 9 \end{bmatrix}$  and that  $Cb = D$ . Solve the matrix equation

$$Ax = b.$$

Q2. Find the value of x and y that satisfy the equations.

$$\begin{bmatrix} 3 & -2 \\ 3 & 0 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y & y \\ x & x \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3y & 3y \\ 10 & 10 \end{bmatrix}$$

Q 3. If,  $E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  and  $F = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$  calculate the matrix product  $EF$  &  $FE$  and show that

$$E^2F + FE^2 = E.$$

Q 4. If A is an orthogonal matrix and  $B = AP$  where P is a non singular matrix then show that the matrix  $PB^{-1}$  is also orthogonal.

Q 5. The matrix,  $R(t)$  is defined by  $R(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$ . Show that,  $R(s)R(t) \equiv R(s+t)$ .

Q 6. Prove that the product of two matrices,  $\begin{bmatrix} \cos^2 \theta & \sin \theta \cos \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}$  &  $\begin{bmatrix} \cos^2 \phi & \sin \phi \cos \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{bmatrix}$  is a null

matrix when  $\theta$  &  $\phi$  differ by an odd multiple of  $\frac{\pi}{2}$ .

Q 7. If,  $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ , then show that the matrix A is a root of the polynomial  $f(x) = x^3 - 6x^2 + 7x + 2$ .

Q.8 For a non zero  $\lambda$ , use induction to prove that : (Only for XII CBSE)

(a) 
$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{bmatrix}, \text{ for every } n \in \mathbb{N}$$

(b) If,  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , then  $(aI + bA)^n = a^n I + na^{n-1} b A$ , where I is a unit matrix of order 2,  $\forall n \in \mathbb{N}$ .

Q9. Find the number of  $2 \times 2$  matrix satisfying

(i)  $a_{ij}$  is 1 or  $-1$  ; (ii)  $a_{11}^2 + a_{12}^2 = a_{21}^2 + a_{22}^2 = 2$  ; (iii)  $a_{11} a_{21} + a_{12} a_{22} = 0$

Q 10. Prove that  $(AB)^T = B^T \cdot A^T$ , where A & B are conformable for the product AB . Also verify the result

for the matrices,  $A = \begin{bmatrix} 1 & 2 \\ 2 & -3 \\ -1 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & -3 & 5 \\ 1 & 2 & 3 \end{bmatrix}$ .

Q 11 Express the matrix  $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & -6 \\ -1 & 0 & 4 \end{bmatrix}$  as a sum of a lower triangular matrix & an upper triangular matrix with zero

in its leading diagonal. Also Express the matrix as a sum of a symmetric & a skew symmetric matrix.

Q 12. Find the inverse of the matrix :

(i)  $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix};$  (ii)  $\begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{bmatrix}$  where w is the cube root of unity.

(iii)  $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

Q 13. Find the matrix A satisfying the matrix equation,  $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$ .

Q 14. A is a square matrix of order n.

$l$  = maximum number of distinct entries if A is a triangular matrix

$m$  = maximum number of distinct entries if A is a diagonal matrix

$p$  = minimum number of zeroes if A is a triangular matrix

If  $l + 5 = p + 2m$ , find the order of the matrix.

Q 15. If A is an idempotent matrix and I is an identity matrix of the same order, find the value of n,  $n \in \mathbb{N}$ , such that  $(A + I)^n = I + 127 A$ .

Q.16 If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  then prove that value of f and g satisfying the matrix equation  $A^2 + fA + gI = O$  are equal to  $-t_r(A)$  and determinant of A respectively. Given a, b, c, d are non zero reals and

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ .

- Q17. Matrices A and B satisfy  $AB = B^{-1}$  where  $B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$ . Find
- (i) without finding  $B^{-1}$ , the value of K for which  $KA - 2B^{-1} + I = \mathbf{O}$
- (ii) Without finding  $A^{-1}$ , the matrix X satisfying  $A^{-1}XA = B$  (iii) the matrix A, using  $A^{-1}$

Q18. For the matrix  $A = \begin{bmatrix} 4 & -4 & 5 \\ -2 & 3 & -3 \\ 3 & -3 & 4 \end{bmatrix}$  find  $A^{-2}$ .

Q19. Given  $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ . Find P such that  $BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Q 20. Use matrix to solve the following system of equations.

(i)  $\begin{matrix} x+y+z=3 \\ x+2y+3z=4 \\ x+4y+9z=6 \end{matrix}$  (ii)  $\begin{matrix} x+y+z=6 \\ x-y+z=2 \\ 2x+y-z=1 \end{matrix}$  (iii)  $\begin{matrix} x+y+z=3 \\ x+2y+3z=4 \\ 2x+3y+4z=7 \end{matrix}$  (iv)  $\begin{matrix} x+y+z=3 \\ x+2y+3z=4 \\ 2x+3y+4z=9 \end{matrix}$

## EXERCISE-5

Q1. Given  $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$ ;  $B = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$ . I is a unit matrix of order 2. Find all possible matrix X in the following cases.

(i)  $AX = A$  (ii)  $XA = I$  (iii)  $XB = \mathbf{O}$  but  $BX \neq \mathbf{O}$ .

Q 2. If A & B are square matrices of the same order & A is symmetrical, show that  $B'AB$  is also symmetrical.

Q 3. Show that,  $\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ .

Q.4 If the matrices  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  and  $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(a, b, c, d not all simultaneously zero) commute, find the value of  $\frac{d-b}{a+c-b}$ . Also show that the

matrix which commutes with A is of the form  $\begin{bmatrix} \alpha - \beta & 2\beta/3 \\ \beta & \alpha \end{bmatrix}$

Q 5. If the matrix A is involutory, show that  $\frac{1}{2}(I + A)$  and  $\frac{1}{2}(I - A)$  are idempotent and

$\frac{1}{2}(I + A) \cdot \frac{1}{2}(I - A) = \mathbf{O}$ .

Q 6. Prove that (i)  $|\text{adj}(\text{adj} A)| = |A|^{(n-1)^2}$ , where A is a non-singular matrix of order 'n'.

(ii)  $\text{adj}(\text{adj} A) = |A|^{n-2} \cdot A$ , where  $|A|$  denotes the determinant of co-efficient matrix.

Q 7. Find the product of two matrices A & B, where  $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$  &  $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$  and use it to

solve the following system of linear equations,

$x + y + 2z = 1$  ;  $3x + 2y + z = 7$  ;  $2x + y + 3z = 2$  .

Q 8. If  $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$  then, find a non-zero square matrix  $X$  of order 2 such that  $AX = \mathbf{O}$ . Is  $XA = \mathbf{O}$ .

If  $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ , is it possible to find a square matrix  $X$  such that  $AX = \mathbf{O}$ . Give reasons for it.

Q 9. If  $A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$ ;  $B = \begin{bmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{bmatrix}$  Where  $0 < \beta < \frac{\pi}{2}$  then prove that  $BAB = A^{-1}$ . Also find the least positive value of  $\alpha$  for which  $BA^4B = A^{-1}$ .

Q 10. If  $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$  is an idempotent matrix. Find the value of  $f(a)$ , where  $f(x) = x - x^2$ , when  $bc = 1/4$ . Hence otherwise evaluate  $a$ .

Q 11. If  $A$  is a skew symmetric matrix and  $I + A$  is non singular, then prove that the matrix  $B = (I - A)(I + A)^{-1}$  is an orthogonal matrix. Use this to find a matrix  $B$  given  $A = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$ .

Q 12. If  $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$  then show that  $F(x) \cdot F(y) = F(x + y)$

Hence prove that  $[F(x)]^{-1} = F(-x)$ .

Q 13. If  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ ;  $B = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ ;  $C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$  and  $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$  then solve the following matrix equation.

(a)  $AX = B - I$       (b)  $(B - I)X = IC$       (c)  $CX = A$

Q 14. Determine the values of  $a$  and  $b$  for which the system  $\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ -1 \end{bmatrix}$

(i) has a unique solution ; (ii) has no solution and (iii) has infinitely many solutions

Q 15. Let  $X$  be the solution set of the equation  $A^x = I$ , where  $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$  and  $I$  is the corresponding

unit matrix and  $x \subseteq \mathbb{N}$  then find the minimum value of  $\sum (\cos^x \theta + \sin^x \theta)$ ,  $\theta \in \mathbb{R}$ .

Q16. Determine the matrices  $B$  and  $C$  with integral element such that

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} = B^3 + C^3$$

Q17. If  $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$  is an orthogonal matrix, find the values of  $\alpha, \beta, \gamma$ .

Q18. If  $A = \begin{bmatrix} k & m \\ l & n \end{bmatrix}$  and  $kn \neq lm$ ; then show that  $A^2 - (k + n)A + (kn - lm)I = \mathbf{O}$ . Hence find  $A^{-1}$ .

Q19. Evaluate  $\lim_{n \rightarrow \infty} \begin{bmatrix} 1 & \frac{x}{n} \\ -\frac{x}{n} & 1 \end{bmatrix}^n$



Q.20 Given matrices  $A = \begin{bmatrix} 1 & x & 1 \\ x & 2 & y \\ 1 & y & 3 \end{bmatrix}$ ;  $B = \begin{bmatrix} 3 & -3 & z \\ -3 & 2 & -3 \\ z & -3 & 1 \end{bmatrix}$

Obtain x, y and z if the matrix AB is symmetric.

## EXERCISE-6

Q.1 If matrix  $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$  where a, b, c are real positive numbers,  $abc = 1$  and  $A^T A = I$ , then find the value of  $a^3 + b^3 + c^3$ . [JEE 2003, Mains-2 out of 60]

Q.2 If  $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$  and  $|A^3| = 125$ , then  $\alpha =$   
 (A)  $\pm 3$  (B)  $\pm 2$  (C)  $\pm 5$  (D) 0 [JEE 2004 (Screening)]

Q.3 If M is a  $3 \times 3$  matrix, where  $M^T M = I$  and  $\det(M) = 1$ , then prove that  $\det(M - I) = 0$ .

Q.4  $A = \begin{bmatrix} a & 0 & 1 \\ 1 & c & b \\ 1 & d & b \end{bmatrix}$ ,  $B = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}$ ,  $U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$ ,  $V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}$ .

If there is vector matrix X, such that  $AX = U$  has infinitely many solution, then prove that  $BX = V$  cannot have a unique solution. If  $afd \neq 0$ , then prove that  $BX = V$  has no solution.

Q.5  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$ ,  $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  and  $A^{-1} = \left[ \frac{1}{6}(A^2 + cA + dI) \right]$ , then the value of c and d are  
 (A) -6, -11 (B) 6, 11 (C) -6, 11 (D) 6, -11

Q.6 If  $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ ,  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  and  $Q = PAP^T$  and  $x = P^T Q^{2005} P$ , then x is equal to

(A)  $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$  (B)  $\begin{bmatrix} 4 + 2005\sqrt{3} & 6015 \\ 2005 & 4 - 2005\sqrt{3} \end{bmatrix}$

(C)  $\frac{1}{4} \begin{bmatrix} 2 + \sqrt{3} & 1 \\ -1 & 2 - \sqrt{3} \end{bmatrix}$  (D)  $\frac{1}{4} \begin{bmatrix} 2005 & 2 - \sqrt{3} \\ 2 + \sqrt{3} & 2005 \end{bmatrix}$

Q7. If f(x) is a quadratic polynomial and a, b, c are three real and distinct numbers satisfying

$\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}$ . Given f(x) cuts the x-axis at A and V is the point of maxima.

If AB is any chord which subtends right angle at V, find curve f(x) and area bounded by chord AB and curve f(x).

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \text{ if } U_1, U_2 \text{ and } U_3 \text{ are columns matrices satisfying.}$$

$$AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, AU_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, AU_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \text{ and } U \text{ is } 3 \times 3 \text{ matrix whose columns are } U_1, U_2, U_3 \text{ then answer the}$$

following questions.

Q8. The value of  $|U|$  is [JEE 2006]

(A) 3 (B) -3 (C) 3/2 (D) 2

Q9. The sum of the elements of  $U^{-1}$  is [JEE 2006]

(A) -1 (B) 0 (C) 1 (D) 3

Q10. The value of  $\begin{bmatrix} 3 & 2 & 0 \end{bmatrix} U \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$  is [JEE 2006]

(A) 5 (B) 5/2 (C) 4 (D) 3/2

## ANSWER SHEET

### EXERCISE-4

Q.1  $x_1 = 1, x_2 = -1, x_3 = 1$       Q.2  $x = \frac{3}{2}, y = 2$       Q.3  $EF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, FE = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Q.9 8      Q.11  $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 5 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & -3 \\ 2 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \\ -3 & 3 & 0 \end{bmatrix}$

Q.12 (i)  $\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix},$  (ii)  $\frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & w^2 & w \\ 1 & w & w^2 \end{bmatrix},$  (iii)  $\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$       Q.13  $\frac{1}{19} \begin{bmatrix} 48 & -25 \\ -70 & 42 \end{bmatrix}$

Q.14 4      Q.15  $n = 7$       Q.16  $f = -(a + d); g = ad - bc$

Q.17 (i)  $K = 2,$  (ii)  $X = B,$  (iii)  $A = \frac{1}{4} \begin{bmatrix} -2 & 2 \\ -4 & 2 \end{bmatrix}$       Q.18  $\begin{bmatrix} 17 & 4 & -19 \\ -10 & 0 & 13 \\ -21 & -3 & 25 \end{bmatrix}$       Q.19  $\begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix}$

Q.20 (i)  $x = 2, y = 1, z = 0;$  (ii)  $x = 1, y = 2, z = 3;$   
 (iii)  $x = 2 + k, y = 1 - 2k, z = k$  where  $k \in \mathbb{R};$  (iv) inconsistent, hence no solution

## EXERCISE-5

Q.1 (i)  $X = \begin{bmatrix} a & b \\ 2-2a & 1-2b \end{bmatrix}$  for  $a, b \in \mathbb{R}$ ; (ii)  $X$  does not exist. ;

(iii)  $X = \begin{bmatrix} a & -3a \\ c & -3c \end{bmatrix}$   $a, c \in \mathbb{R}$  and  $3a + c \neq 0$ ;  $3b + d \neq 0$

Q.4 1      Q.7  $x = 2, y = 1, z = -1$       Q.8  $X = \begin{bmatrix} -2c & -2d \\ c & d \end{bmatrix}$ , where  $c, d \in \mathbb{R} - \{0\}$ , NO

Q.9  $\frac{2\pi}{3}$       Q.10  $f(a) = 1/4, a = 1/2$       Q.11  $\frac{1}{13} \begin{bmatrix} -12 & -5 \\ 5 & -12 \end{bmatrix}$

Q.13(a)  $X = \begin{bmatrix} -3 & -3 \\ 5 & 2 \\ 2 & 2 \end{bmatrix}$ , (b)  $X = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$ , (c) no solution

Q.14 (i)  $a \neq -3, b \in \mathbb{R}$ ; (ii)  $a = -3$  and  $b \neq 1/3$ ; (iii)  $a = -3, b = 1/3$

Q.15 2      Q.16  $B = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$  and  $C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Q.17  $\alpha = \pm \frac{1}{\sqrt{2}}, \beta = \pm \frac{1}{\sqrt{6}}, \gamma = \pm \frac{1}{\sqrt{3}}$       Q.18  $\frac{1}{kn - lm} \begin{bmatrix} n & -m \\ -l & k \end{bmatrix}$

Q.19  $\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$

Q.20  $\left(-\frac{4\sqrt{2}}{3}, \frac{2}{3}, 2\sqrt{2}\right), \left(\frac{4\sqrt{2}}{3}, \frac{2}{3}, -2\sqrt{2}\right), (3, 3, -1)$

## EXERCISE-6

Q.1 4      Q.2 A      Q.5 C      Q.6 A      Q.7.  $\frac{125}{3}$  sq. units      Q.8 A

Q.9. B      Q.10. A