5. VECTOR

In physics we deal with two type of physical quantity one is scalar and other is vector. Each scalar quantities has magnitude.

Magnitude of a physical quantity means product of numerical value and unit of that physical quantity.

For example mass = 4 kg

Magnitude of mass = 4 kg and unit of mass = kg

Example of scalar quantities : mass, speed, distance etc.

Scalar quantities can be added, subtracted and multiplied by simple laws of algebra.

5.1 DEFINITION OF VECTOR

If a physical quantity in addition to magnitude -

(a) has a specified direction.

(b) It should obey commutative law of additions $\vec{A}+\vec{B}=\vec{B}+\vec{A}$

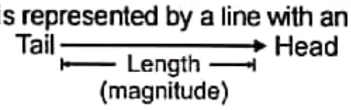
(c) obeys the law of parallelogram of addition, then and then only it is said to be a vector. If any of the above conditions is not satisfied the physical quantity cannot be a vector.

If a physical quantity is a vector it has a direction, but the converse may or may not be true, i.e. if a physical quantity has a direction, it may or may not a be vector. example : pressure, surface tension or current etc. have directions but are not vectors because they do not obey parallelogram law of addition.

The magnitude of a vector ($\vec{A}$) is the absolute value of a vector and is indicated by $|\vec{A}|$ or A.
Example of vector quantity : Displacement, velocity, acceleration, force etc.

Representation of vector :

Geometrically, the vector is represented by a line with an arrow indicating the direction of vector as

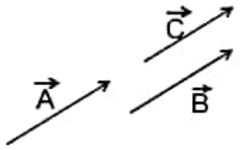


Mathematically, vector is represented by $\vec{A}$

Sometimes it is represented by bold letter **A** .

IMPORTANT POINTS :

☞ If a vector is displaced parallel to itself it does not change (see Figure)



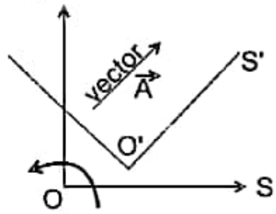
$\vec{A} = \vec{B} = \vec{C}$
Transition of a vector parallel to itself

☞ If a vector is rotated through an angle other than multiple

$\vec{A} \neq \vec{B}$ Rotation of a vector

of 2π (or 360^o) it changes (see Figure).

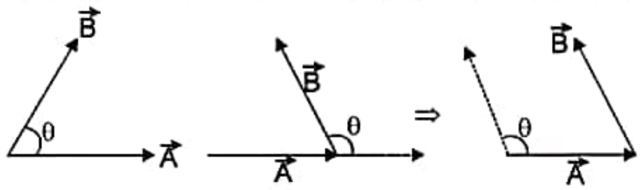
☞ If the frame of reference is translated or rotated the vector does not change (though its components may change). (see Figure).



moving frame

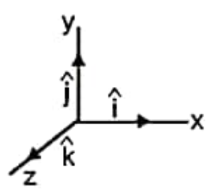
☞ Two vectors are called equal if their magnitudes and directions are same, and they represent values of same physical quantity.

☞ Angle between two vectors means smaller of the two angles between the vectors when they are placed tail to tail by displacing either of the vectors parallel to itself (i.e. $0 \leq \theta \leq \pi$).



5.2 UNIT VECTOR

Unit vector is a vector which has a unit magnitude and points in a particular direction. Any vector $(\vec{A})$ can be written as the product of unit vector $(\hat{A})$ in that direction and magnitude of the given vector.

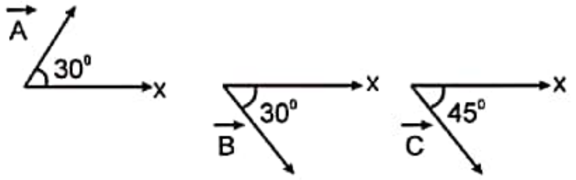


$\vec{A} = A\hat{A}$ or $\hat{A} = \frac{\vec{A}}{A}$

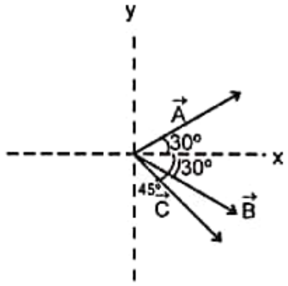
A unit vector has no dimensions and unit. Unit vectors along the positive x-, y- and z-axes of a rectangular coordinate system are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively such that $|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$.

Solved Examples

Example 57. Three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ are shown in the figure. Find angle between (i) $\vec{A}$ and $\vec{B}$, (ii) $\vec{B}$ and $\vec{C}$, (iii) $\vec{A}$ and $\vec{C}$.



Solution : To find the angle between two vectors we connect the tails of the two vectors. We can shift $\vec{B}$ such that tails of $\vec{A}$, $\vec{B}$ and $\vec{C}$ are connected as shown in figure.



Now we can easily observe that angle between $\vec{A}$ and $\vec{B}$ is 60°, $\vec{B}$ and $\vec{C}$ is 15° and between $\vec{A}$ and $\vec{C}$ is 75°.

Example 58. A unit vector along East is defined as $\hat{i}$. A force of 10^5 dynes acts west wards. Represent the force in terms of $\hat{i}$.

Solution : $\vec{F} = -10^5\hat{i}$ dynes

5.3 MULTIPLICATION OF A VECTOR BY A SCALAR

Multiplying a vector $\vec{A}$ with a positive number λ gives a vector $\vec{B}(=\lambda\vec{A})$ whose magnitude is changed by the factor λ but the direction is the same as that of $\vec{A}$. Multiplying a vector $\vec{A}$ by a negative number λ gives a vector $\vec{B}$ whose direction is opposite to the direction of $\vec{A}$ and whose magnitude is $-\lambda$ times .

Solved Examples

Example 59. A physical quantity (m = 3kg) is multiplied by a vector $\vec{a}$ such that $\vec{F} = m\vec{a}$. Find the magnitude and direction of $\vec{F}$ if

(i) $\vec{a} = 3\text{m/s}^2$ East wards

(ii) $\vec{a} = -4\text{m/s}^2$ North wards

Solution : (i) $\vec{F} = m\vec{a} = 3 \times 3\ \text{ms}^{-2}$ East wards = 9 N East wards

(ii) $\vec{F} = m\vec{a} = 3 \times (-4)$ N North wards = – 12N North wards = 12 N South wards

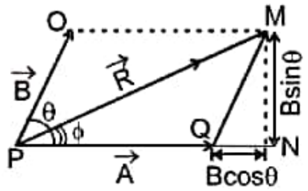
5.4 ADDITION OF VECTORS

Addition of vectors is done by parallelogram law or the triangle law :

(a) Parallelogram law of addition of vectors : If two vectors $\vec{A}$ and $\vec{B}$ are represented by two adjacent sides of a parallelogram both pointing outwards (and their tails coinciding) as shown. Then the diagonal drawn through the intersection of the two vectors represents the resultant (i.e., vector sum of $\vec{A}$ and $\vec{B}$).

$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

The direction of resultant vector from $\vec{R}$ is $\vec{A}$ given by

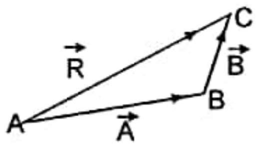


$$\tan\phi = \frac{MN}{PN} = \frac{MN}{PQ + QN} = \frac{B\sin\theta}{A + B\cos\theta}$$

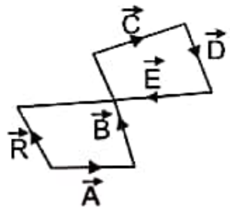
$$\phi = \tan^{-1}\left(\frac{B\sin\theta}{A + B\cos\theta}\right)$$

(b) Triangle law of addition of vectors : To add two vectors $\vec{A}$ and $\vec{B}$ shift any of the two vectors parallel to itself until the tail of $\vec{B}$ is at the head of $\vec{A}$. The sum $\vec{A} + \vec{B}$ is a vector $\vec{R}$ drawn from the tail of $\vec{A}$ to the head of $\vec{B}$, i.e., $\vec{A} + \vec{B} = \vec{R}$. As the figure formed is a triangle, this method is called ' triangle method' of addition of vectors.

If the 'triangle method' is extended to add any number of vectors in one operation as shown. Then the figure formed is a polygon and hence the name Polygon Law of addition of vectors is given to such type of addition.



$\vec{A} + \vec{B} = \vec{R}$



$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E}$

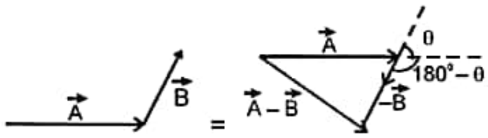
IMPORTANT POINTS :

- To a vector only a vector of same type can be added that represents the same physical quantity and the resultant is a vector of the same type.
- As $R = [A^2 + B^2 + 2AB\cos\theta]^{1/2}$ so R will be maximum when, $\cos\theta$ = max = 1, i.e., $\theta = 0°$, i.e. vectors are like or parallel and $R_{max} = A + B$.
- The resultant will be minimum if, $\cos\theta$ = min = -1, i.e., $\theta = 180°$, i.e. vectors are antiparallel and $R_{min} = A \sim B$.
- If the vectors A and B are orthogonal, i.e., $\theta = 90°$, $R = \sqrt{A^2 + B^2}$
- As previously mentioned that the resultant of two vectors can have any value from $(A \sim B)$ to $(A + B)$ depending on the angle between them and the magnitude of resultant decreases as θ increases 0° to 180°
- Minimum number of unequal coplanar vectors whose sum can be zero is three.
- The resultant of three non-coplanar vectors can never be zero, or minimum number of non coplanar vectors whose sum can be zero is four.

☞ Subtraction of a vector from a vector is the addition of negative vector, i.e.,

$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

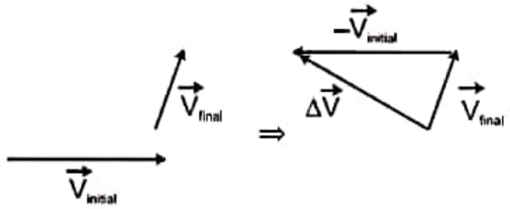
(a) From figure it is clear $\vec{A} - \vec{B}$ that is equal to addition of $\vec{A}$ with reverse of $\vec{B}$



$|\vec{A} - \vec{B}| = [(A)^2 + (B^2) + 2AB\cos(180° - \theta)]^{1/2}$

$|\vec{A} - \vec{B}| = \sqrt{A^2 + B^2 - 2AB\cos\theta}$

(b) Change in a vector physical quantity means subtraction of initial vector from the final vector.



Solved Examples

Example 60. Find the resultant of two forces each having magnitude F_0, and angle between them is θ.

Solution : $F^2_{Resultant} = F_0^2 + F_0^2 + 2F_0^2 \cos\theta$

$= 2F_0^2 (1 + \cos\theta) = 2F_0^2 (1 + 2\cos^2\frac{\theta}{2} - 1) = 2F_0^2 \times 2\cos^2\frac{\theta}{2}$

$F_{resultant} = 2F_0 \cos\frac{\theta}{2}$

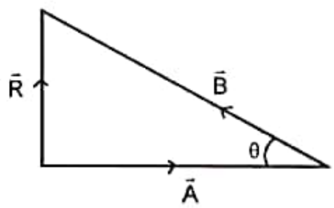
Example 61. Two non zero vectors $\vec{A}$ and $\vec{B}$ are such that $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$. Find angle between $\vec{A}$ and $\vec{B}$?

Solution : $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}| \Rightarrow A^2 + B^2 + 2AB\cos\theta = A^2 + B^2 - 2AB\cos\theta$

$\Rightarrow 4AB\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

Example 62. The resultant of two velocity vectors $\vec{A}$ and $\vec{B}$ is perpendicular to $\vec{A}$. Magnitude of Resultant is $\vec{R}$ equal to half magnitude of $\vec{B}$. Find the angle between $\vec{A}$ and $\vec{B}$?

Solution : Since $\vec{R}$ is perpendicular to $\vec{A}$. Figure shows the three vectors $\vec{A}$, $\vec{B}$ and $\vec{R}$ .angle between $\vec{A}$ and $\vec{B}$ is $\pi - \theta$



$\sin\theta = \frac{R}{B} = \frac{B}{2B} = \frac{1}{2} \Rightarrow \theta = 30°$

⇒ angle between A and B is 150°.

Example 63. If the sum of two unit vectors is also a unit vector. Find the magnitude of their difference?

Solution : Let $\hat{A}$ and $\hat{B}$ are the given unit vectors and $\hat{R}$ is their resultant then $|\hat{R}| = |\hat{A} + \hat{B}|$

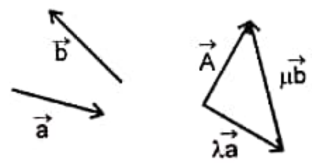
$1 = \sqrt{(\hat{A})^2 + (\hat{B})^2 + 2|\hat{A}||\hat{B}|\cos\theta}$

$1 = 1 + 1 + 2\cos\theta \Rightarrow \cos\theta = -\frac{1}{2}$

$|\vec{A} - \vec{B}| = \sqrt{(\hat{A})^2 + (\hat{B})^2 - 2|\hat{A}||\hat{B}|\cos\theta} = \sqrt{1 + 1 - 2\times1\times1(-\frac{1}{2})} = \sqrt{3}$

5.5 RESOLUTION OF VECTORS

If $\vec{a}$ and $\vec{b}$ be any two nonzero vectors in a plane with different directions and $\vec{A}$ be another vector in the same plane. $\vec{A}$ can be expressed as a sum of two vectors - one obtained by multiplying $\vec{a}$ by a real number and the other obtained by multiplying $\vec{b}$ by another real number.



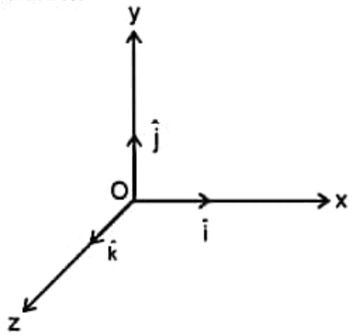
$\vec{A} = \lambda\vec{a} + \mu\vec{b}$ (where λ and μ are real numbers)

We say that $\vec{A}$ has been resolved into two component vectors namely $\lambda\vec{a}$ and $\mu\vec{b}$

$\lambda\vec{a}$ and $\mu\vec{b}$ along $\vec{a}$ and $\vec{b}$ respectively. Hence one can resolve a given vector into two component vectors along a set of two vectors – all the three lie in the same plane.

Resolution along rectangular component :

It is convenient to resolve a general vector along axes of a rectangular coordinate system using vectors of unit magnitude, which we call as unit vectors. $\hat{i}, \hat{j}, \hat{k}$ are unit vector along x, y and z-axis as shown in figure below:



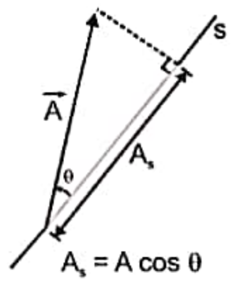
Resolution in two Dimension

Consider a vector $\vec{A}$ that lies in xy plane as shown in figure,

$\vec{A} = \vec{A}_1 + \vec{A}_2$

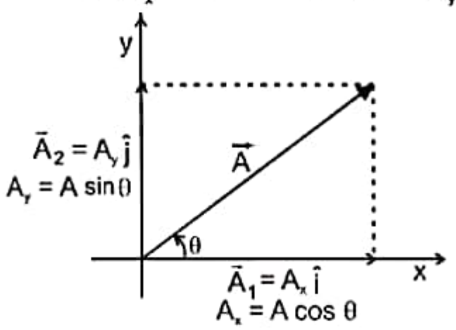
$\vec{A}_1 = A_x\hat{i}$, $\vec{A}_2 = A_y\hat{j}$

$\Rightarrow \vec{A} = A_x\hat{i} + A_y\hat{j}$



The quantities A_x and A_y are called x- and y-components of the vector.

A_x is itself not a vector but $A_x\hat{i}$ is a vector and so is $A_y\hat{j}$



$A_x = A\cos\theta$ and $A_y = A\sin\theta$

Its clear from above equation that a component of a vector can be positive, negative or zero depending on the value of θ. A vector $\vec{A}$ can be specified in a plane by two ways :

(a) its magnitude A and the direction θ it makes with the x-axis; or

(b) its components A_x and A_y. $A = \sqrt{A_x^2 + A_y^2}$, $\theta = \tan^{-1}\dfrac{A_y}{A_x}$

Note : If $A = A_x \Rightarrow A_y = 0$ and if $A = A_y \Rightarrow A_x = 0$ i.e. components of a vector perpendicular to itself is always zero.

The rectangular components of each vector and those of the sum $\vec{C} = \vec{A} + \vec{B}$ are shown in figure. We saw that

$\vec{C} = \vec{A} + \vec{B}$ is equivalent to both $C_x = A_x + B_x$ and $C_y = A_y + B_y$

Resolution in three dimensions. A vector $\vec{A}$ in components along x-, y- and z-axis can be written as :

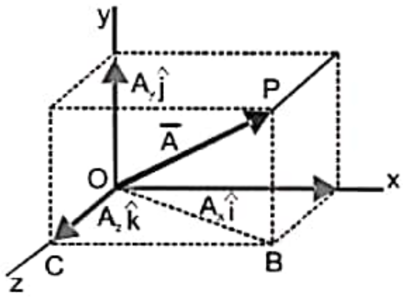
$\vec{OP} = \vec{OB} + \vec{BP} = \vec{OC} + \vec{CB} + \vec{BP}$

$\Rightarrow \vec{A} = \vec{A}_z + \vec{A}_x + \vec{A}_y = \vec{A}_x + \vec{A}_y + \vec{A}_z = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

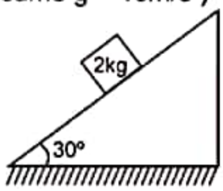
$A_x = A\cos\alpha$, $A_y = A\cos\beta$, $A_z = A\cos\gamma$

where $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are termed as **Direction Cosines** of a given vector $\vec{A}$.$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

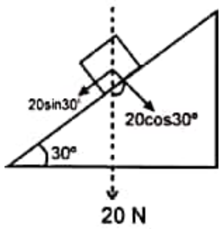


Solved Examples

Example 64. A mass of 2 kg lies on an inclined plane as shown in figure. Resolve its weight along and perpendicular to the plane.(Assume $g = 10 m/s^2$)



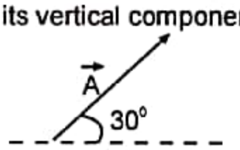
Solution : Component along the plane



$= 20 \sin 30 = 10\sqrt{3}$ N.

Component perpendicular to the plane = 20 cos 30 = 10N

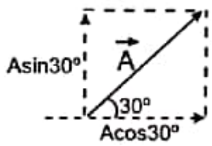
Example 65. A vector makes an angle of 30° with the horizontal. If horizontal component of the vector is 250. Find magnitude of vector and its vertical component ?



Solution : Let vector is $\vec{A}$

$A_x = A\cos 30^0 = 250 = \dfrac{A\sqrt{3}}{2} \Rightarrow A = \dfrac{500}{\sqrt{3}}$

$A_y = A\sin 30^0 = \dfrac{500}{\sqrt{3}} \times \dfrac{1}{2} = \dfrac{250}{\sqrt{3}}$



Example 66. $\vec{A} = \hat{i} + 2\hat{j} - 3\hat{k}$, when a vector $\vec{B}$ is added to $\vec{A}$, we get a unit vector along x-axis. Find the value of $\vec{B}$? Also find its magnitude

Solution : $\vec{A} + \vec{B} = \hat{i}$

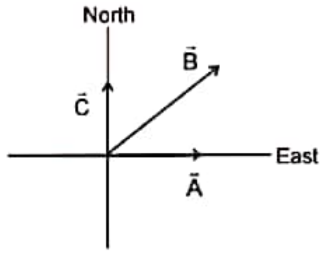
$\vec{B} = \hat{i} - \vec{A} = \hat{i} - (\hat{i} + 2\hat{j} - 3\hat{k}) = -2\hat{j} + 3\hat{k} \Rightarrow |\vec{B}| = \sqrt{(2)^2 + (3)^2} = \sqrt{13}$

Example 67. In the above question find a unit vector along $\vec{B}$?

Solution : $\hat{B} = \frac{\vec{B}}{B} = \frac{-2\hat{j} + 3\hat{k}}{\sqrt{13}}$

Example 68. Vector $\vec{A}$, $\vec{B}$ and $\vec{C}$ have magnitude 5, $5\sqrt{2}$ and 5 respectively, direction of $\vec{A}$, $\vec{B}$ and $\vec{C}$ are towards east, North-East and North respectively. If $\hat{i}$ and $\hat{j}$ are unit vectors along East and North respectively. Express the sum $\vec{A} + \vec{B} + \vec{C}$ in terms of $\hat{i}$, $\hat{j}$. Also find magnitude and direction of the resultant.

Solution : $\vec{A} = 5\hat{i}$ $\qquad \vec{C} = 5\hat{j}$



$\vec{B} = 5\sqrt{2}\cos 45\hat{i} + 5\sqrt{2}\sin 45\hat{j} = 5\hat{i} + 5\hat{j}$

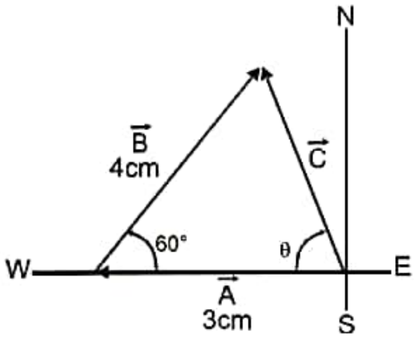
$\vec{A} + \vec{B} + \vec{C} = 5\hat{i} + 5\hat{i} + 5\hat{j} + 5\hat{j} = 10\hat{i} + 10\hat{j}$

$|\vec{A} + \vec{B} + \vec{C}| = \sqrt{(10)^2 + (10)^2} = 10\sqrt{2}$

$\tan\theta = \frac{10}{10} = 1 \quad \Rightarrow \quad \theta = 45^\circ$ from East

Example 69. You walk 3 Km west and then 4 Km headed 60° north of east. Find your resultant displacement
(a) graphically and (b) using vector components.

Solution : Picture the Problem : The triangle formed by the three vectors is not a right triangle, so the magnitudes of the vectors are not related by the Pythagoras theorem. We find the resultant graphically by drawing each of the displacements to scale and measuring the resultant displacement.



(a) If we draw the first displacement vector 3 cm long and the second one 4 cm long, we find the resultant vector to be about 3.5 cm long. Thus the magnitude of the resultant displacement is 3.5 Km. The angle θ made between the resultant displacement and the west direction can then be measured with a protractor. It is about 75°.

(b) 1. Let $\vec{A}$ be the first displacement and choose the x-axis to be in the easterly direction. Compute A_x and A_y, $A_x = -3$, $A_y = 0$

2. Similarly, compute the components of the second displacement $\vec{B}$, $B_x = 4\cos 60^\circ = 2$, $B_y = 4\sin 60^\circ = 2\sqrt{3}$

3. The components of the resultant displacement $\vec{C} = \vec{A} + \vec{B}$ are found by addition,

$\vec{C} = (-3+2)\hat{i} + (2\sqrt{3})\hat{j} = -\hat{i} + 2\sqrt{3}\hat{j}$

4. The Pythagorean theorem gives the magnitude of $\vec{C}$.

$C = \sqrt{1^2 + (2\sqrt{3})^2} = \sqrt{13} = 3.6$

5. The ratio of C_y to C_x gives the tangent of the angle θ between $\vec{C}$ and the x axis.

$\tan\theta = \frac{2\sqrt{3}}{-1} \Rightarrow \theta = -74°$

Remark : Since the displacement (which is a vector) was asked for, the answer must include either the magnitude and direction, or both components. in (b) we could have stopped at step 3 because the x and y components completely define the displacement vector. We converted to the magnitude and direction to compare with the answer to part (a). Note that in step 5 of (b), a calculator gives the angle as –74°. But the calculator can't distinguish whether the x or y components is negative. We noted on the figure that the resultant displacement makes an angle of about 75° with the negative x axis and an angle of about 105° with the positive x axis. This agrees with the results in (a) within the accuracy of our measurement.

5.6 MULTIPLICATION OF VECTORS

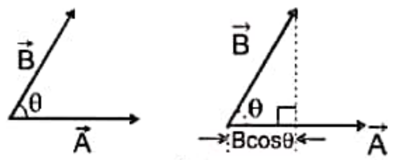
5.6.1 THE SCALAR PRODUCT

The scalar product or dot product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A}.\vec{B}$ (read $\vec{A}$ dot $\vec{B}$) is defined as the product of their magnitude with cosine of angle between them. Thus, $\vec{A}.\vec{B} = AB\cos\theta$ {here θ is the angle between the vectors}

PROPERTIES :

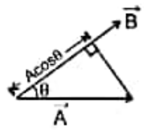
- It is always a scalar which is positive if angle between the vectors is acute (i.e. < 90°) and negative if angle between them is obtuse (i.e. $90° < \theta \le 180°$)
- It is commutative, i.e., $\vec{A}.\vec{B} = \vec{B}.\vec{A}$
- It is distributive, i.e. $\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$
- As by definition $\vec{A}.\vec{B} = AB\cos\theta$. The angle between the vectors $\theta = \cos^{-1}\left[\frac{\vec{A}.\vec{B}}{AB}\right]$
- $\vec{A}.\vec{B} = A(B\cos\theta) = B(A\cos\theta)$

  Geometrically, B cos θ is the projection of $\vec{B}$ onto $\vec{A}$ and A cos θ is the projection of $\vec{A}$ onto $\vec{B}$ as shown. So $\vec{A}.\vec{B}$ is the product of the magnitude of $\vec{A}$ and the component of $\vec{B}$ along $\vec{A}$ and vice versa.



Component of $\vec{B}$ along $\vec{A} = B\cos\theta = \frac{\vec{A}.\vec{B}}{A} = \hat{A}.\vec{B}$

Component of $\vec{A}$ along $\vec{B} = A\cos\theta = \frac{\vec{A}.\vec{B}}{B} = \vec{A}.\hat{B}$



- Scalar product of two vectors will be maximum when cos θ = max = 1, i.e., θ = 0°, i.e., vectors are parallel $\Rightarrow (\vec{A}.\vec{B})_{max} = AB$
- If the scalar product of two nonzero vectors vanishes then the vectors are perpendicular.
- The scalar product of a vector by itself is termed as self dot product and is given by
  $(\vec{A})^2 = \vec{A}.\vec{A} = AA\cos\theta = AA\cos 0^\circ = A^2 \Rightarrow \quad A = \sqrt{\vec{A}.\vec{A}}$
- In case of unit vector $\hat{n}$,
  $\hat{n}.\hat{n} = 1 \times 1 \times \cos 0^\circ = 1 \qquad \Rightarrow \quad \hat{n}.\hat{n} = \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$
- In case of orthogonal unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$; $\quad \hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$
  $\vec{A}.\vec{B} = (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z).(\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) = [A_xB_x + A_yB_y + A_zB_z]$

Solved Examples

Example 70. If the Vectors $\vec{P} = a\hat{i} + a\hat{j} + 3\hat{k}$ and $\vec{Q} = a\hat{i} - 2\hat{j} - \hat{k}$ are perpendicular to each other. Find the value of a?

Solution : If vectors $\vec{P}$ and $\vec{Q}$ are perpendicular

$\Rightarrow \vec{P}\cdot\vec{Q} = 0 \quad \Rightarrow (a\hat{i} + a\hat{j} + 3\hat{k}).(a\hat{i} - 2\hat{j} - \hat{k}) = 0 \quad \Rightarrow a^2 - 2a - 3 = 0$

$\Rightarrow a^2 - 3a + a - 3 = 0 \Rightarrow a(a-3) + 1(a-3) = 0 \quad \Rightarrow a = -1, 3$

Example 71. Find the component of $3\hat{i} + 4\hat{j}$ along $\hat{i} + \hat{j}$?

Solution : Componant of $\vec{A}$ along $\vec{B}$ is given by $\frac{\vec{A}\cdot\vec{B}}{B}$ hence required component

$= \frac{(3\hat{i}+4\hat{j})\cdot(\hat{i}+\hat{j})}{\sqrt{2}} = \frac{7}{\sqrt{2}}$

Example 72. Find angle between $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = 12\hat{i} + 5\hat{j}$?

Solution : We have $\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{(3\hat{i}+4\hat{j})\cdot(12\hat{i}+5\hat{j})}{\sqrt{3^2+4^2}\sqrt{12^2+5^2}}$

$\cos\theta = \frac{36+20}{5\times 13} = \frac{56}{65} \qquad \theta = \cos^{-1}\frac{56}{65}$

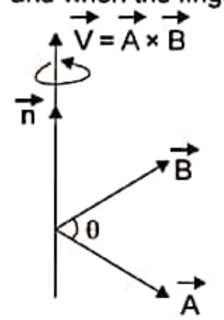
5.6.2 VECTOR PRODUCT

The vector product or cross product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A}\times\vec{B}$ (read $\vec{A}$ cross $\vec{B}$) is defined as : $\vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$.

Here θ is the angle between the vectors and the direction $\hat{n}$ is given by the right-hand-thumb rule.

Right-Hand-Thumb Rule:

To find the direction of $\hat{n}$, draw the two vectors $\vec{A}$ and $\vec{B}$ with both the tails coinciding. Now place your stretched right palm perpendicular to the plane of $\vec{A}$ and $\vec{B}$ in such a way that the fingers are along the vector $\vec{A}$ and when the fingers are closed they go towards $\vec{B}$.



The direction of the thumb gives the direction of $\hat{n}$.

PROPERTIES :

- Vector product of two vectors is always a vector perpendicular to the plane containing the two vectors i.e. orthogonal to both the vectors $\vec{A}$ and $\vec{B}$, though the vectors $\vec{A}$ and $\vec{B}$ may or may not be orthogonal.
- Vector product of two vectors is not commutative i.e. $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$.

  But $\left|\vec{A}\times\vec{B}\right| = \left|\vec{B}\times\vec{A}\right| = AB\sin\theta$
- The vector product is distributive when the order of the vectors is strictly maintained i.e.

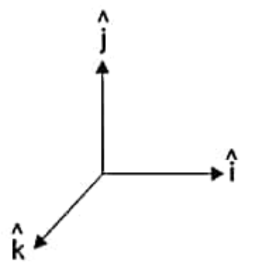
  $A\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$.
- The magnitude of vector product of two vectors will be maximum when sinθ = max = 1, i.e, θ = 90°. $|\vec{A}\times\vec{B}|_{max} = AB$ i.e., magnitude of vector product is maximum if the vectors are orthogonal.
- The magnitude of vector product of two non–zero vectors will be minimum when |sinθ| = minimum = 0, i.e., θ = 0° or 180° and $|\vec{A}\times\vec{B}|_{min} = 0$ i.e., if the vector product of two non–zero vectors vanishes, the vectors are collinear.

  Note : When θ = 0° then vectors may be called as like vector or parallel vectors and when θ = 180° then vectors may be called as unlike vectors or antiparallel vectors.
- The self cross product i.e. product of a vector by itself vanishes i.e. is a null vector.

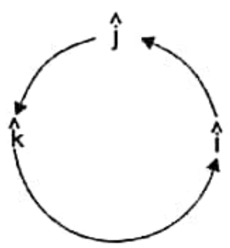
  Note : Null vector or zero vector : A vector of zero magnitude is called zero vector. The direction of a zero vector is in determinate (unspecified). $\vec{A}\times\vec{A} = AA\sin 0^\circ\ \hat{n} = \vec{0}$.

  In case of unit vector $\hat{n}$, $\hat{n}\times\hat{n} = \vec{0} \Rightarrow \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$
- In case of orthogonal unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ in accordance with right-hand-thumb-rule,

  $\hat{i}\times\hat{j} = \hat{k}$ $\hat{j}\times\hat{k} = \hat{i}$ $\hat{k}\times\hat{i} = \hat{j}$



(A)



(B)

- In terms of components,

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$\vec{A}\times\vec{B} = \hat{i}(A_yB_z - A_zB_y) + \hat{j}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)$$

- The magnitude of area of the parallelogram formed by the adjacent sides of vectors $\vec{A}$ and $\vec{B}$ equal to $|\vec{A}\times\vec{B}|$

Solved Examples

Example 73. $\vec{A}$ is Eastwards and $\vec{B}$ is downwards. Find the direction of $\vec{A} \times \vec{B}$?

Solution : Applying right hand thumb rule we find that $\vec{A} \times \vec{B}$ is along North.

Example 74. If $\vec{A} \cdot \vec{B} = |\vec{A} \times \vec{B}|$, find angle between $\vec{A}$ and $\vec{B}$

Solution : $\vec{A} \cdot \vec{B} = |\vec{A} \times \vec{B}|$ AB cos θ = AB sin θ   tan θ = 1   ⇒ θ = 45°

Example 75. Two vectors $\vec{A}$ and $\vec{B}$ are inclined to each other at an angle θ. Find a unit vector which is perpendicular to both $\vec{A}$ and $\vec{B}$

Solution : $\vec{A} \times \vec{B} = AB \sin\theta\,\hat{n}$ ⇒ $\hat{n} = \dfrac{\vec{A}\times\vec{B}}{AB\sin\theta}$ here $\hat{n}$ is perpendicular to both $\vec{A}$ and $\vec{B}$.

Example 76 Find $\vec{A} \times \vec{B}$ if $\vec{A} = \hat{i} - 2\hat{j} + 4\hat{k}$ and $\vec{B} = 2\hat{i} - \hat{j} + 2\hat{k}$.

Solution : $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 4 \\ 3 & -1 & 2 \end{vmatrix} = \hat{i}\,(-4-(-4)) - \hat{j}\,(2-12) + \hat{k}\,(-1-(-6)) = 10\hat{j} + 5\hat{k}$

Miscellaneous Problems

Problem 1. Find the value of

(a) sin (– θ)   (b) cos (– θ)   (c) tan (– θ)   (d) $\cos(\frac{\pi}{2} - \theta)$

(e) $\sin(\frac{\pi}{2} + \theta)$ (f) $\cos(\frac{\pi}{2} + \theta)$ (g) sin (π – θ)   (h) cos (π – θ)

(i) $\sin(\frac{3\pi}{2} - \theta)$ (j) $\cos(\frac{3\pi}{2} - \theta)$ (k) $\sin(\frac{3\pi}{2} + \theta)$ (l) $\cos(\frac{3\pi}{2} + \theta)$

(m) $\tan(\frac{\pi}{2} - \theta)$ (n) $\cot(\frac{\pi}{2} - \theta)$

Answers :
(a) – sin θ   (b) cos θ   (c) – tan θ   (d) sin θ
(e) cos θ   (f) – sin θ   (g) sin θ   (h) – cos θ
(i) – cos θ   (j) – sin θ   (k) – cos θ   (l) sin θ
(m) cot θ   (n) tan θ

Problem 2. (i) For what value of m the vector $\vec{A} = 2\hat{i} + 3\hat{j} - 6\hat{k}$ is perpendicular to $\vec{B} = 3\hat{i} - m\hat{j} + 6\hat{k}$

(ii) Find the components of vector $\vec{A} = 2\hat{i} + 3\hat{j}$ along the direction of $\hat{i} + \hat{j}$?

Answers : (i) m = –10   (ii) $\frac{5}{\sqrt{2}}$.

Problem 3. (i) $\vec{A}$ is North–East and $\vec{B}$ is down wards, find the direction of $\vec{A} \times \vec{B}$.

(ii) Find $\vec{B} \times \vec{A}$ if $\vec{A} = 3\hat{i} - 2\hat{j} + 6\hat{k}$ and $\vec{B} = \hat{i} - \hat{j} + \hat{k}$.

Answers : (i) North - West.   (ii) $-4\hat{i} - 3\hat{j} + \hat{k}$