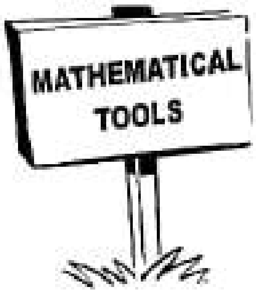
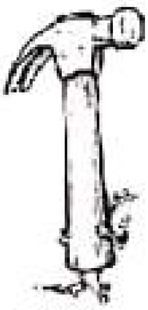
# MATHEMATICAL TOOLS

Mathematics is the language of physics. It becomes easier to describe, understand and apply the physical principles, if one has a good knowledge of mathematics.



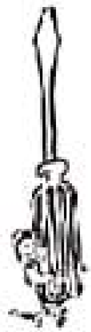

**• Tools are required to do physical work easily and mathematical tools are required to solve numerical problems easily.**

**MATHEMATICAL TOOLS**



Differentiation

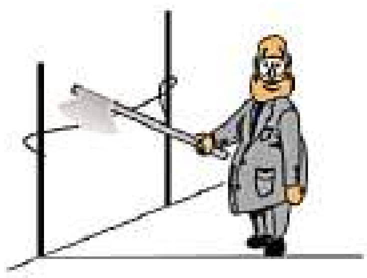
Integration



Vectors

**To solve the problems of physics Newton made significant contributions to Mathematics by inventing differentiation and integration.**

Cutting a tree with a blade



Cutting a string with an axe

# 1. FUNCTION

Function is a rule of relationship between two variables in which one is assumed to be dependent and the other independent variable, for example :

**e.g.** The temperatures at which water boils depends on the elevation above sea level (the boiling point drops as you ascend). Here elevation above sea level is the independent & temperature is the dependent variable

**e.g.** The interest paid on a cash investment depends on the length of time the investment is held. Here time is the independent and interest is the dependent variable.

In each of the above example, value of one variable quantity (dependent variable) , which we might call y, depends on the value of another variable quantity (independent variable), which we might call x. Since the value of y is completely determined by the value of x, we say that y is a function of x and represent it mathematically as $y = f(x)$.

Here f represents the function, x the independent variable & y is the dependent variable.


All possible values of independent variables (x) are called **domain** of function.

All possible values of dependent variable (y) are called **range** of function.

Think of a function f as a kind of machine that produces an output value f(x) in its range whenever we feed it an input value x from its domain (figure).

When we study circles, we usually call the area A and the radius r. Since area depends on radius, we say that A is a function of r, $A = f(r)$ . The equation $A = \pi r^2$ is a rule that tells how to calculate a unique (single) output value of A for each possible input value of the radius r.

$A = f(r) = \pi r^2$ . (Here the rule of relationship which describes the function may be described as square & multiply by $\pi$).

If $r = 1$ $A = \pi$ ; if $r = 2$ $A = 4\pi$ ; if $r = 3$ $A = 9\pi$

The set of all possible input values for the radius is called the domain of the function. The set of all output values of the area is the range of the function.

We usually denote functions in one of the two ways :

1. By giving a formula such as $y = x^2$ that uses a dependent variable y to denote the value of the function.
2. By giving a formula such as $f(x) = x^2$ that defines a function symbol f to name the function.

   Strictly speaking, we should call the function f and not f(x),
   y = sin x. Here the function is sine, x is the independent variable.

## Solved Examples

**Example 1.** The volume V of a ball (solid sphere) of radius r is given by the function $V(r) = (4/3)\pi(r)^3$ . The volume of a ball of radius 3m is ?

**Solution :** $V(3) = 4/3\pi(3)^3 = 36\pi\ m^3$ .

**Example 2.** Suppose that the function F is defined for all real numbers r by the formula $F(r) = 2(r - 1) + 3$. Evaluate F at the input values 0, 2, x + 2, and F(2).

**Solution :** In each case we substitute the given input value for r into the formula for F :

$F(0) = 2(0 - 1) + 3 = -2 + 3 = 1$ ;

$F(2) = 2(2 - 1) + 3 = 2 + 3 = 5$

$F(x + 2) = 2(x + 2 - 1) + 3 = 2x + 5$ ;

$F(F(2)) = F(5) = 2(5 - 1) + 3 = 11$.

**Example 3.** A function $f(x)$ is defined as $f(x) = x^2 + 3$. Find $f(0)$, $f(1)$, $f(x^2)$, $f(x+1)$ and $f(f(1))$.

**Solution :** $f(0) = 0^2 + 3 = 3$ ; $f(1) = 1^2 + 3 = 4$ ; $f(x^2) = (x^2)^2+3 = x^4+3$

$f(x+1) = (x + 1)^2 + 3 = x^2 + 2x + 4$ ; $f(f(1)) = f(4) = 4^2+3 = 19$

**Example 4.** If function F is defined for all real numbers x by the formula $F(x) = x^2$. Evaluate F at the input values 0, 2, x + 2 and F(2)

**Solution :** $F(0) = 0$ ; $F(2) = 2^2 = 4$ ; $F(x+2) = (x+2)^2$

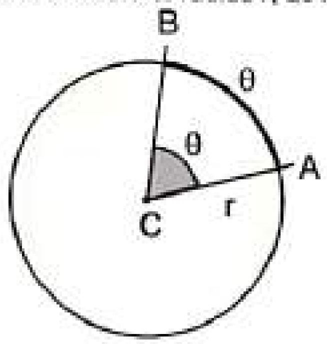
$F(F(2)) = F(4) = 4^2 = 16$

# 2. TRIGONOMETRY

## 2.1 MEASUREMENT OF ANGLE AND RELATIONSHIP BETWEEN DEGREES AND RADIAN

In navigation and astronomy, angles are measured in degrees, but in calculus it is best to use units called radians because of the way they simplify later calculations.

Let ACB be a central angle in a **circle** of radius r, as in figure.




Then the angle ACB or θ is defined in radius as $\theta = \dfrac{\text{Arc length}}{\text{Radius}} \Rightarrow \theta = \dfrac{\overset{\frown}{AB}}{r}$

If r = 1 then $\theta = AB$

The **radian measure** for a circle of unit radius of angle ACB is defined to be the length of the circular arc AB. Since the circumference of the circle is $2\pi$ and one complete revolution of a circle is 360°, the relation between radians and degrees is given by : $\pi$ radians = 180°

**Angle Conversion formulas**

1 degree = $\dfrac{\pi}{180}$ ($\approx 0.02$) radian

1 radian $\dfrac{\pi}{180}$ = 57 degrees

| Degrees to radians : multiply by $\dfrac{\pi}{180}$ |
|---|
| Radians to degrees : multiply by $\dfrac{180}{\pi}$ |

## Solved Examples

**Example 5.** (i) Convert 45° to radians. (ii) Convert $\dfrac{\pi}{6}$ rad to degrees.

**Solution :** (i) $45 \cdot \dfrac{\pi}{180} = \dfrac{\pi}{4}$ rad (ii) $\dfrac{\pi}{6} \cdot \dfrac{180}{\pi} = 30°$

**Example 6.** Convert 30° to radians.

**Solution :** $30° \times \dfrac{\pi}{180} = \dfrac{\pi}{6}$ rad

**Example 7.** Convert $\dfrac{\pi}{3}$ rad to degrees.

**Solution :** $\dfrac{\pi}{3} \times \dfrac{180}{\pi} = 60°$

**Standard values**

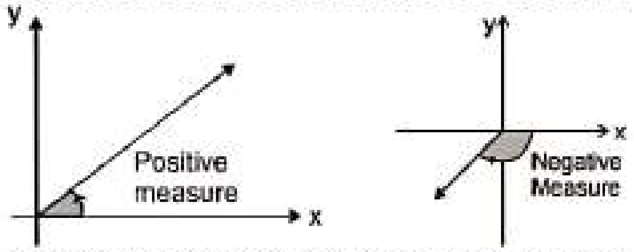
(1) $30^0 = \frac{\pi}{6}$ rad (2) $45^\circ = \frac{\pi}{4}$ rad (3) $60^\circ$ = rad

(4) $90^\circ = \frac{\pi}{2}$ rad (5) $120^\circ = \frac{2\pi}{3}$ rad (6) $135^\circ$ = rad

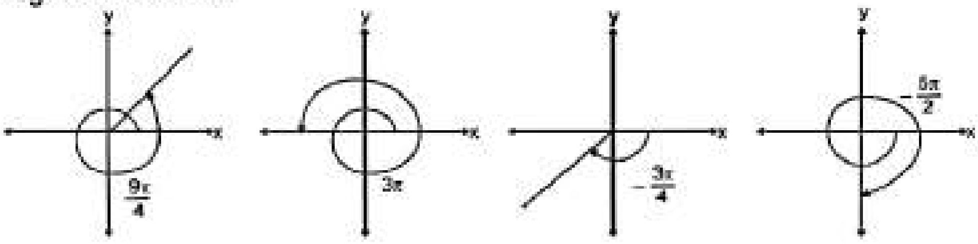
(7) $150^\circ = \frac{5\pi}{6}$ rad (8) $180^\circ = \pi$ rad (9) $360^\circ = 2\pi$ rad

(Check these values yourself to see that they satisfy the conversion formulae)

## 2.2. MEASUREMENT OF POSITIVE AND NEGATIVE ANGLES

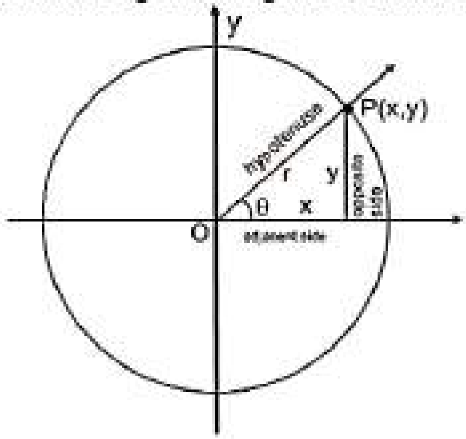



An angle in the xy-plane is said to be in standard position if its vertex lies at the origin and its initial ray lies along the positive x-axis (Fig.). Angles measured counterclockwise from the positive x-axis are assigned positive measures ; angles measured clockwise are assigned negative measures.




## 2.3 SIX BASIC TRIGONOMETRIC FUNCTIONS

The trigonometric function of a general angle θ are defined in terms of x, y, and r.




Sine : $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$ Cosecant : $\text{cosec}\theta = \frac{\text{hyp}}{\text{opp}} = \frac{r}{y}$

Cosine : $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$ Secant : $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{r}{x}$

Tangent : $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$ Cotangent : $\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{x}{y}$

**VALUES OF TRIGONOMETRIC FUNCTIONS**

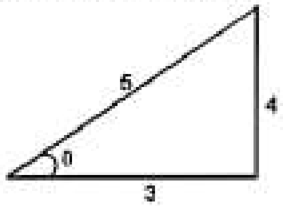
If the circle in (Fig. above) has radius r = 1, the equations defining $\sin\theta$ and $\cos\theta$ become

$\cos\theta = x$, $\sin\theta = y$

We can then calculate the values of the cosine and sine directly from the coordinates of P.

## *Solved Examples*

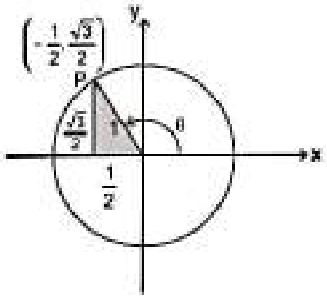
**Example 8.** Find the six trigonometric ratios from given figure




**Solution :** $\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{4}{5}$ ; $\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{3}{5}$; $\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{4}{3}$ ; $\text{cosec}\,\theta = \dfrac{\text{hyp}}{\text{opp}} = \dfrac{5}{4}$ ;

$\sec\theta = \dfrac{\text{hyp}}{\text{adj}} = \dfrac{5}{3}$ ; $\cot\theta = \dfrac{\text{adj}}{\text{opp}} = \dfrac{3}{4}$

**Example 9.** Find the sine and cosine of angle θ shown in the unit circle if coordinate of point p are as shown.




**Solution :** $\cos\theta$ = x-coordinate of P = $-\dfrac{1}{2}$; $\sin\theta$ = y-coordinate of P = $\dfrac{\sqrt{3}}{2}$.

## 2.4 RULES FOR FINDING TRIGONOMETRIC RATIO OF ANGLES GREATER THAN 90°

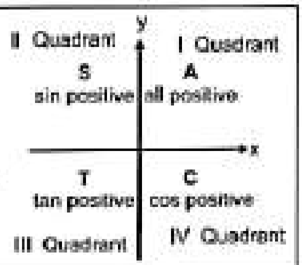
**Step 1 →** Identify the quadrant in which angle lies.

**Step 2 →**

**(a)** If angle = $(n\pi \pm \theta)$ where n is an integer. Then trigonometric function of $(n\pi \pm \theta)$ = same trigonometric function of θ and sign will be decided by CAST Rule.

**THE CAST RULE**

A useful rule for remembering when the basic trigonometric functions are positive and negative is the CAST rule.
If you are not very enthusiastic about CAST, You can remember it as ASTC (After school to college)



(b) If angle = $\left[(2n+1)\frac{\pi}{2} \pm \theta\right]$ where n is an integer. Then trigonometric function of $\left[(2n+1)\frac{\pi}{2} \pm \theta\right]$

= complimentary trignometric function of $\theta$ and sign will be decided by CAST Rule.

| Degree | 0 | 30 | 37 | 45 | 53 | 60 | 90 | 120 | 135 | 180 |
|---|---|---|---|---|---|---|---|---|---|---|
| Radians | 0 | $\pi/6$ | $37\pi/180$ | $\pi/4$ | $53\pi/180$ | $\pi/3$ | $\pi/2$ | $2\pi/3$ | $3\pi/4$ | $\pi$ |
| $\sin\theta$ | 0 | 1/2 | 3/5 | $1/\sqrt{2}$ | 4/5 | $\sqrt{3}/2$ | 1 | $\sqrt{3}/2$ | $1/\sqrt{2}$ | 0 |
| $\cos\theta$ | 1 | $\sqrt{3}/2$ | 4/5 | $1/\sqrt{2}$ | 3/5 | 1/2 | 0 | $-1/2$ | $-1/\sqrt{2}$ | $-1$ |
| $\tan\theta$ | 0 | $1/\sqrt{3}$ | 3/4 | 1 | 4/3 | $\sqrt{3}$ | $\infty$ | $-\sqrt{3}$ | $-1$ | 0 |

**Values of sin θ, cos θ and tan θ for some standard angles.**

## *Solved Examples*

**Example 10.** Evaluate sin 120°

**Solution :** $\sin 120° = \sin(90° + 30°) = \cos 30° = \frac{\sqrt{3}}{2}$

**Aliter** $\sin 120° = \sin(180° - 60°) = \sin 60° = \frac{\sqrt{3}}{2}$

**Example 11.** Evaluate cos 135°

**Solution :** $\cos 135° = \cos(90° + 45°) = -\sin 45° = -\frac{1}{\sqrt{2}}$

**Example 12.** Evaluate cos 210°

**Solution :** $\cos 210° = \cos(180° + 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$

**Example 13.** Evaluate tan 210°

**Solution :** $\tan 210° = \tan(180° + 30°) = \tan 30° = \frac{1}{\sqrt{3}}$

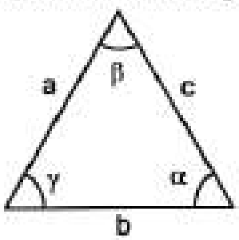
## 2.5 GENERAL TRIGONOMETRIC FORMULAS :

1. $\cos^2\theta + \sin^2\theta = 1$
   $1 + \tan^2\theta = \sec^2\theta$
   $1 + \cot^2\theta = \text{cosec}^2\theta$

2. $\cos(A+B) = \cos A \cos B - \sin A \sin B$
   $\sin(A+B) = \sin A \cos B + \cos A \sin B$
   $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

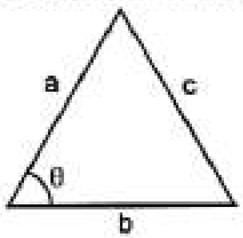
3. $\sin 2\theta = 2\sin\theta\cos\theta$ ; $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
   $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ ; $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

4. **sine rule for triangles**




$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

5. **cosine rule for triangles**




$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

# 3. DIFFERENTIATION

## 3.1 FINITE DIFFERENCE

The finite difference between two values of a physical quantity is represented by $\Delta$ notation.

For example :

Difference in two values of y is written as $\Delta y$ as given in the table below.

| | | | |
|---|---|---|---|
| $y_2$ | 100 | 100 | 100 |
| $y_1$ | 50 | 99 | 99.5 |
| $\Delta y = y_2 - y_1$ | 50 | 1 | 0.5 |

**INFINITELY SMALL DIFFERENCE :**

The infinitely small difference means very-very small difference. And this difference is represented by 'd' notation instead of '$\Delta$'.

For example infinitely small difference in the values of y is written as 'dy'

if $y_2 = 100$ and $y_1 = 99.99999999$........

then dy = 0.000000..................00001

## 3.2 DEFINITION OF DIFFERENTIATION

Another name for differentiation is derivative. Suppose y is a function of x or y = f(x).

Differentiation of y with respect to x is denoted by symbol f '(x) where $f'(x) = \frac{dy}{dx}$

dx is very small change in x and dy is corresponding very small change in y.

**NOTATION :** There are many ways to denote the derivative of a function y = f(x). Besides f '(x), the most common notations are these :

| | | |
|---|---|---|
| $y'$ | "y prime" or "y dash" | Nice and brief but does not name the independent variable. |
| $\frac{dy}{dx}$ | "dy by dx" | Names the variables and uses d for derivative. |
| $\frac{df}{dx}$ | "df by dx" | Emphasizes the function's name. |
| $\frac{d}{dx}f(x)$ | "d by dx of f" | Emphasizes the idea that differentiation is an operation performed on f. |
| $D_xf$ | "dx of f" | A common operator notation. |
| $\dot{y}$ | "y dot" | One of Newton's notations, now common for time derivatives i.e. $\frac{dy}{dt}$. |
| $f'(x)$ | f dash x | Most common notation, it names the independent variable and Emphasize the function's name. |

## 3.3 SLOPE OF A LINE

It is the tan of angle made by a line with the positive direction of x-axis, measured in anticlockwise direction.

Slope = $\tan\theta$ (In 1st quadrant $\tan\theta$ is +ve & 2nd quadrant $\tan\theta$ is –ve )

In Figure - 1 slope is positive

$\theta < 90°$ (1st quadrant)

In Figure - 2 slope is negative

$\theta > 90°$ (2nd quadrant)

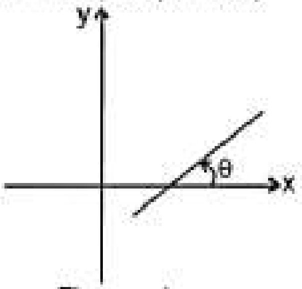



Figure - 1

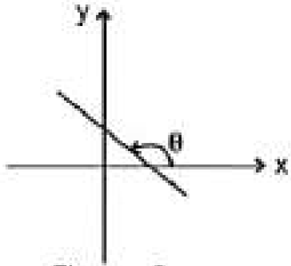


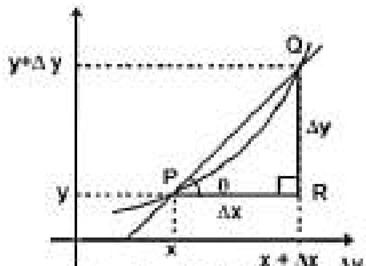

Figure - 2

## 3.4 AVERAGE RATES OF CHANGE :

Given an arbitrary function y = f(x) we calculate the average rate of change of y with respect to x over the interval (x, x + $\Delta x$) by dividing the change in value of y, i.e. $\Delta y = f(x + \Delta x) - f(x)$, by length of interval $\Delta x$ over which the change occurred.

The average rate of change of y with respect to x over the interval [x, x + $\Delta x$]

$$= \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$




In triangle QPR $\tan\theta = \frac{\Delta y}{\Delta x}$

Geometrically, $\frac{\Delta y}{\Delta x} = \frac{QR}{PR} = \tan\theta$ = Slope of the line PQ

therefore we can say that average rate of change of y with respect to x is equal to slope of the line joining P & Q.

## 3.5 THE DERIVATIVE OF A FUNCTION

We know that, average rate of change of y w.r.t. x is $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$. If the limit of this ratio exists as $\Delta x \to 0$, then it is called the derivative of given function f(x) and is denoted as

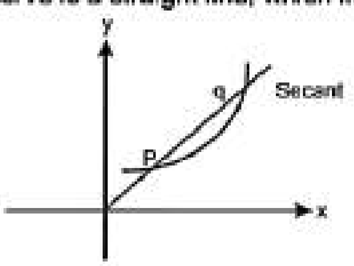
$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

## 3.6 GEOMETRICAL MEANING OF DIFFERENTIATION

The geometrical meaning of differentiation is very much useful in the analysis of graphs in physics. To understand the geometrical meaning of derivatives we should have knowledge of secant and tangent to a curve.

**Secant and tangent to a curve**

**Secant :** A secant to a curve is a straight line, which intersects the curve at any two points.



**Tangent :** A tangent is a straight line, which touches the curve at a particular point. Tangent is a limiting case of secant which intersects the curve at two overlapping points.

In the figure-1 shown, if value of $\Delta x$ is gradually reduced then the point Q will move nearer to the point P. If the process is continuously repeated (Figure - 2) value of $\Delta x$ will be infinitely small and secant PQ to the given curve will become a tangent at point P.

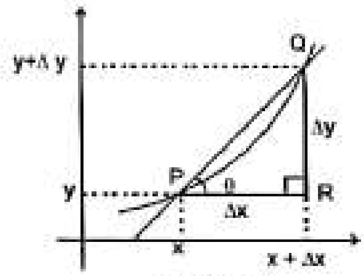



Figure - 1

Therefore $\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \frac{dy}{dx} = \tan\theta$

we can say that differentiation of y with respect to x, i.e. $\left(\frac{dy}{dx}\right)$ is equal to slope of the tangent at point P (x, y) or $\tan\theta = \frac{dy}{dx}$ (From fig. 1, the average rate of change of y from x to x + $\Delta x$ is identical with the slope of secant PQ.)

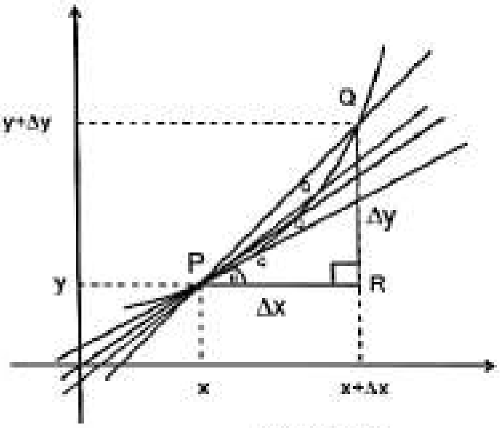



Figure - 2

## 3.7 RULES FOR DIFFERENTIATION

### RULE NO. 1 : DERIVATIVE OF A CONSTANT

The first rule of differentiation is that the derivative of every constant function is zero.

If c is constant, then $\frac{d}{dx}c = 0$.

**Example 14.** $\frac{d}{dx}(8) = 0,\ \frac{d}{dx}\left(-\frac{1}{2}\right) = 0,\ \frac{d}{dx}\left(\sqrt{3}\right) = 0$

### RULE NO. 2 : POWER RULE

If n is a real number, then $\frac{d}{dx}x^n = nx^{n-1}$.

To apply the power Rule, we subtract 1 from the original exponent (n) and multiply the result by n.

**Example 15.**

| $f$ | $x$ | $x^2$ | $x^3$ | $x^4$ | .... |
|---|---|---|---|---|---|
| $f'$ | $1$ | $2x$ | $3x^2$ | $4x^3$ | .... |

**Example 16.** (i) $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$

(ii) $\frac{d}{dx}\left(\frac{4}{x^3}\right) = 4\frac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\frac{12}{x^4}$.

**Example 17.** (a) $\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Function defined for $x \geq 0$ — derivative defined only for $x > 0$

(b) $\frac{d}{dx}(x^{1/5}) = \frac{1}{5}x^{-4/5}$

Function defined for $x \geq 0$ — derivative not defined at $x = 0$

**RULE NO. 3 : THE CONSTANT MULTIPLE RULE**

If u is a differentiable function of x, and c is a constant, then $\frac{d}{dx}(cu) = c\frac{du}{dx}$

In particular, if n is a positive integer, then $\frac{d}{dx}(cx^n) = cn\,x^{n-1}$

**Example 18.** The derivative formula $\frac{d}{dx}(3x^2) = 3(2x) = 6x$ says that if we rescale the graph of $y = x^2$ by multiplying each y–coordinate by 3, then we multiply the slope at each point by 3.

**Example 19.** A useful special case. The derivative of the negative of a differentiable function is the negative of the function's derivative. Rule 3 with $c = -1$ gives.

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1\cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{d}{dx}(u)$$

**RULE NO. 4 : THE SUM RULE**

The derivative of the sum of two differentiable functions is the sum of their derivatives.

If u and v are differentiable functions of x, then their sum u + v is differentiable at every point where u and v are both differentiable functions is their derivatives.

$$\frac{d}{dx}(u-v) = \frac{d}{dx}[u+(-1)v] = \frac{du}{dx} + (-1)\frac{dv}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

The Sum Rule also extends to sums of more than two functions, as long as there are only finitely many functions in the sum. If $u_1$, $u_2$,.........$u_n$ are differentiable at x, then so is $u_1 + u_2 + \ldots\ldots + u_n$, and $\frac{d}{dx}(u_1 + u_2 + \ldots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \ldots\ldots + \frac{du_n}{dx}$.

**Example 20.** (a) $y = x^4 + 12x$ (b) $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) \qquad \frac{dy}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

$$= 4x^3 + 12 \qquad = 3x^2 + \frac{4}{3}.2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5.$$

Notice that we can differentiate any polynomial term by term, the way we differentiated the polynomials in above example.

### RULE NO. 5 : THE PRODUCT RULE

If u and v are differentiable at x, then so is their product **uv**, and $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

The derivative of the product **uv** is u times the derivative of **v** plus v times the derivative of u.

In prime notation $(uv)' = uv' + vu'$.

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is not the product of their derivatives. For instance,

$\frac{d}{dx}(x.x) = \frac{d}{dx}(x^2) = 2x$, while $\frac{d}{dx}(x).\ \frac{d}{dx}(x) = 1.1 = 1.$

**Example 21.** Find the derivatives of $y = \frac{4}{3}(x^2 + 1)(x^3 + 3)$.

**Solution :** From the product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\frac{d}{dx}[(x^2+1)(x^3+3)] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x)$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x = 5x^4 + 3x^2 + 6x.$$

Example can be done as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial. We now check : $y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

This is in agreement with our first calculation. There are times, however, when the product Rule must be used. In the following examples. We have only numerical values to work with.

**Example 22.** Let y = uv be the product of the functions u and v. Find $y'(2)$ if $u(2) = 3$, $u'(2) = -4$, $v(2) = 1$, and $v'(2) = 2$.

**Solution :** From the Product Rule, in the form $y' = (uv)' = uv' + vu'$,

we have $y'(2) = u(2)\,v'(2) + v(2)\,u'(2) = (3)(2) + (1)(-4) = 6 - 4 = 2.$

### RULE NO. 6 : THE QUOTIENT RULE

If u and v are differentiable at x, and $v(x) \neq 0$, then the quotient u/v is differentiable at x,

and $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of the quotient of two functions is not the quotient of their derivatives.

**Example 23.** Find the derivative of $y = \frac{t^2 - 1}{t^2 + 1}$

**Solution :** We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^2 + 1$ :

$$\frac{dy}{dt} = \frac{(t^2+1).2t - (t^2-1).2t}{(t^2+1)^2} \Rightarrow \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2} = \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2} = \frac{4t}{(t^2+1)^2}.$$

**RULE NO. 7 : DERIVATIVE OF SINE FUNCTION**

$$\frac{d}{dx}(\sin x) = \cos x$$

**Example 24.** (a) $y = x^2 - \sin x$ : $\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x)$ Difference Rule

$= 2x - \cos x$

(b) $y = x^2 \sin x$ : $\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + 2x \sin x$ Product Rule

$= x^2 \cos x + 2x \sin x$

(c) $y = \frac{\sin x}{x}$ : $\frac{dy}{dx} = \frac{x.\frac{d}{dx}(\sin x) - \sin x.1}{x^2}$ Quotient Rule

$= \frac{x\cos x - \sin x}{x^2}.$

**RULE NO. 8 : DERIVATIVE OF COSINE FUNCTION**

$$\frac{d}{dx}(\cos x) = -\sin x$$

**Example 25.** (a) $y = 5x + \cos x$

$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x)$ Sum Rule

$= 5 - \sin x$

(b) $y = \sin x \cos x$

$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x(\sin x) \frac{d}{dx}$ Product Rule

$= \sin x(-\sin x) + \cos x(\cos x) = \cos^2 x - \sin^2 x$

**RULE NO. 9 : DERIVATIVES OF OTHER TRIGONOMETRIC FUNCTIONS**

Because sin x and cos x are differentiable functions of x , the related functions

$\tan x = \frac{\sin x}{\cos x}$; $\sec x = \frac{1}{\cos x}$

$\cot x = \frac{\cos x}{\sin x}$ ; $\operatorname{cosec} x = \frac{1}{\sin x}$

are differentiable at every value of x at which they are defined. There derivatives. Calculated from the Quotient Rule, are given by the following formulas.

$\frac{d}{dx}(\tan x) = \sec^2 x$ ; $\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ ; $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

**Example 26.** Find dy / dx if y = tan x .

**Solution :** $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$

$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

**Example 27.** (a) $\frac{d}{dx}(3x + \cot x) = 3 + \frac{d}{dx}(\cot x) = 3 - \text{cosec}^2 x$

(b) $\frac{d}{dx}\left(\frac{2}{\sin x}\right) = \frac{d}{dx}(2\text{cosec } x) = 2\frac{d}{dx}(\text{cosec } x) = 2(-\text{cosec } x \cot x) = -2 \text{ cosec } x \cot x$

**RULE NO. 10 : DERIVATIVE OF LOGARITHM AND EXPONENTIAL FUNCTIONS**

$\frac{d}{dx}(\log_e x) = \frac{1}{x} \Rightarrow \frac{d}{dx}(e^x) = e^x$

**Example 28.** $y = e^x . \log_e(x)$

$\frac{dy}{dx} = \frac{d}{dx}(e^x) . \log(x) + . \frac{d}{dx}[\log_e(x)] e^x \Rightarrow \frac{dy}{dx} = e^x . \log_e(x) + \frac{e^x}{x}$

**RULE NO. 11 : CHAIN RULE OR "OUTSIDE INSIDE" RULE**

$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

It sometimes helps to think about the Chain Rule the following way. If y = f(g(x)),

$\frac{dy}{dx} = f'[g(x)].g'(x)$ .

In words : To find dy/dx , differentiate the "outside" function f and leave the "inside" g(x) alone ; then multiply by the derivative of the inside.

We now know how to differentiate sin x and $x^2 - 4$, but how do we differentiate a composite like $\sin(x^2 - 4)$? The answer is, with the Chain Rule, which says that the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points. The Chain Rule is probably the most widely used differentiation rule in mathematics. This section describes the rule and how to use it. We begin with examples.

## *Solved Examples*

**Example 29.** The function y = 6x – 10 = 2(3x – 5) is the composite of the functions y = 2u and u = 3x – 5. How are the derivatives of these three functions related ?

**Solution :** We have $\frac{dy}{dx} = 6$, $\frac{dy}{du} = 2$, $\frac{du}{dx} = 3$ .

Since 6 = 2.3, $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

Is it an accident that $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$ ?

If we think of the derivative as a rate of change, our intuition allows us to see that this relationship is reasonable. For y = f(u) and u = g(x) , if y changes twice as fast as u and u changes three times as fast as x, then we expect y to change six times as fast as x.

**Example 30.** We sometimes have to use the Chain Rule two or more times to find a derivative. Here is an example. Find the derivative of $g(t) = \tan(5 - \sin 2t)$

**Solution :** $g'(t) = \dfrac{d}{dt}(\tan(5 - \sin 2t)$

$= \sec^2(5 - \sin 2t) \cdot \dfrac{d}{dt}(5 - \sin 2t)$

Derivative of
tan u with
$u = 5 - \sin 2t$

Derivative of
$5 - \sin u$
with $u = 2t$

$= \sec^2(5 - \sin 2t) \cdot (0 - (\cos 2t) \cdot \dfrac{d}{dt}(2t)$

$= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2$

$= -2(\cos 2t)\sec^2(5 - \sin 2t)$

**Example 31.** (a) $\dfrac{d}{dx}(1 - x^2)^{1/4} = \dfrac{1}{4}(1 - x^2)^{-3/4}(-2x)$ $\qquad u = 1 - x^2$ and $n = 1/4$

Function defined
on $[-1, 1]$

$= \dfrac{-x}{2(1 - x^2)^{3/4}}$

derivative defined
only on $(-1, 1)$

(b) $\dfrac{d}{dx}\sin 2x = \cos 2x \dfrac{d}{dx}(2x) = \cos 2x \cdot 2 = 2\cos 2x$

(c) $\dfrac{d}{dt}(A\sin(\omega t + \phi)$

$= A\cos(\omega t + \phi)\dfrac{d}{dt}(\omega t + \phi) = A\cos(\omega t + \phi).\ \omega. = A\,\omega\cos(\omega t + \phi)$

**RULE NO. 12 : POWER CHAIN RULE**

If u(x) is a differentiable function and where n is a Real number, then $u^n$ is differentiable and

$\dfrac{d}{dx}u^n = nu^{n-1}\dfrac{du}{dx}, \forall n \in R$

**Example 32.** (a) $\dfrac{d}{dx}\sin^5 x = 5\sin^4 x \dfrac{d}{dx}(\sin x) = 5\sin^4 x \cos x$

(b) $\dfrac{d}{dx}(2x + 1)^{-3} = -3(2x + 1)^{-4}\dfrac{d}{dx}(2x + 1) = -3(2x + 1)^{-4}(2) = -6(2x + 1)^{-4}$

(c) $\dfrac{d}{dx}\left(\dfrac{1}{3x - 2}\right) = \dfrac{d}{dx}(3x - 2)^{-1} = -1(3x - 2)^{-2}(3x - 2)\dfrac{d}{dx} = -1(3x - 2)^{-2}(3) = -\dfrac{3}{(3x - 2)^2}$

In part (c) we could also have found the derivative with the Quotient Rule.

**Example 33.** Find the value of $\frac{d}{dx}$ $(Ax + B)^n$

**Solution :** Here u = Ax + B, = $\frac{du}{dx}$ A

$\therefore \frac{d}{dx} (Ax + B)^n = n(Ax + B)^{n-1} . A$

**RULE NO. 13 : RADIAN VS. DEGREES**

$$\frac{d}{dx}\sin(x^\circ) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x^\circ).$$

## 3.8 DOUBLE DIFFERENTIATION

If f is differentiable function, then its derivative f ' is also a function, so f ' may have a derivative of its own, denoted by (f ' )' = f " . This new function f " is called the second derivative of f because it is the derivative of the derivative of f. Using Leibniz notation, we write the second derivative of y = f (x) as

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$. Another notation is $f''(x) = D_2f(x) = D^2f(x)$

**INTERPRETATION OF DOUBLE DERIVATIVE**

We can interpret f''(x) as the slope of the curve y = f'(x) at the point (x, f'(x)). In other words, it is the rate of change of the slope of the original curve y = f(x).

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is acceleration, which we define as follows.

If s = s(t) is the position function of an object that moves in a straight line, we known that its first derivative represents the velocity v(t) of the object as a function of time : $v(t) = s'(t) = \frac{ds}{dt}$

The instantaneous rate of change of velocity with respect to time is called the acceleration a(t) of the object. Thus, the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function : a(t) = v'(t) = s''(t) or in Leibniz notation,

$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

## *Solved Examples*

**Example 34.** If f (x) = x cos x, find f''(x).

**Solution :** Using the Product Rule, we have $f'(x) = x\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(x)$

$= -x \sin x + \cos x$

To find f " (x) we differentiate f' (x) : $f''(x) = \frac{d}{dx}(-x\sin x + \cos x)$

$= -x\frac{d}{dx}(\sin x) + \sin x\ \frac{d}{dx}(-x) + \frac{d}{dx}(\cos x) = -x\cos x - \sin x - \sin x = -x\cos x - 2\sin x$

**Example 35.** The position of a particle is given by the equation $s = f(t) = t^3 - 6t^2 + 9t$ where t is measured in seconds and s in meters. Find the acceleration at time t. What is the acceleration after 4s ?

**Solution :** The velocity function is the derivative of the position function : $s = f(t) = t^3 - 6t^2 + 9t$

$\Rightarrow$ $v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$

The acceleration is the derivative of the velocity function : $a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt} = 6t - 12$

$\Rightarrow$ $a(4) = 6(4) - 12 = 12\ m/s^2$

## 3.9 APPLICATION OF DERIVATIVES

### 3.9.1 DIFFERENTIATION AS A RATE OF CHANGE

$\frac{dy}{dx}$ is rate of change of 'y' with respect to 'x' :

For examples :

(i) $v = \frac{dx}{dt}$ this means velocity 'v' is rate of change of displacement 'x' with respect to time 't'

(ii) $a = \frac{dv}{dt}$ this means acceleration 'a' is rate of change of velocity 'v' with respect to time 't' .

(iii) $F = \frac{dp}{dt}$ this means force 'F' is rate of change of momentum 'p' with respect to time 't' .

(iv) $\tau = \frac{dL}{dt}$ this means torque '$\tau$' is rate of change of angular momentum 'L' with respect to time 't'

(v) Power = $\frac{dW}{dt}$ this means power 'P' is rate of change of work 'W' with respect to time 't'

(vi) $I = \frac{dq}{dt}$ this means current 'I' is rate of flow of charge 'q' with respect to time 't'

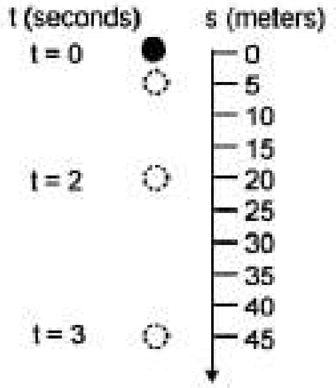
## *Solved Examples*

**Example 36.** The area A of a circle is related to its diameter by the equation $A = \frac{\pi}{4}D^2$. How fast is the area changing with respect to the diameter when the diameter is 10m ?

**Solution :** The (instantaneous) rate of change of the area with respect to the diameter is $\frac{dA}{dD} = \frac{\pi}{4}2D = \frac{\pi D}{2}$.

When D = 10 m, the area is changing at rate $(\pi/2)\ 10 = 5\pi\ m^2/m$. This means that a small change $\Delta D$ m in the diameter would result in a change of about $5\pi\Delta D m^2$ in the area of the circle.

**Example 37.** Experimental and theoretical investigations revealed that the distance a body released from rest falls in time t is proportional to the square of the amount of time it has fallen. We express this by saying that




A ball bearing falling from rest

$$s = \frac{1}{2}gt^2,$$

where s is distance and g is the acceleration due to Earth's gravity. This equation holds in a vacuum, where there is no air resistance, but it closely models the fall of dense, heavy objects in air. Figure shows the free fall of a heavy ball bearing released from rest at time t = 0 sec.

(a) How many meters does the ball fall in the first 2 sec?

(b) What is its velocity, speed, and acceleration then?

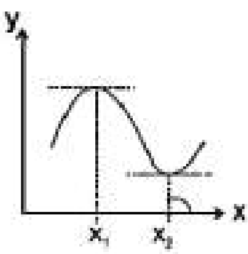
**Solution :** (a) The free–fall equation is $s = 4.9\,t^2$. During the first 2 sec. the ball falls $s(2) = 4.9(2)^2 = 19.6$ m,

(b) At any time t, velocity is derivative of displacement : $v(t) = s'(t) = \frac{d}{dt}(4.9t^2) = 9.8\,t$.

At t = 2, the velocity is v(2) = 19.6 m/sec in the downward (increasing s) direction. The speed at t = 2 is speed = |v(2)| = 19.6 m/sec. $a = \frac{d^2s}{dt^2} = 9.8\ m/s^2$

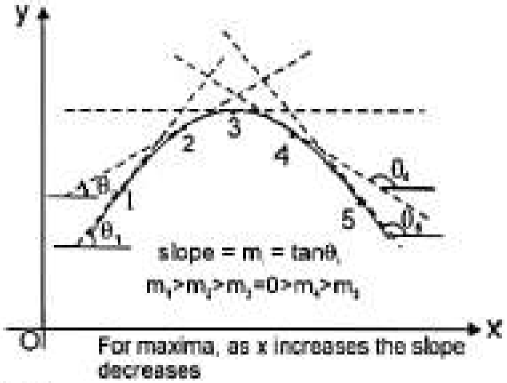
### 3.9.2 MAXIMA AND MINIMA

Suppose a quantity y depends on another quantity x in a manner shown in the figure. It becomes maximum at $x_1$ and minimum at $x_2$. At these points the tangent to the curve is parallel to the x-axis and hence its slope is $\tan\theta = 0$. Thus, at a maximum or a minimum, slope $= \frac{dy}{dx} = 0$.




**MAXIMA**

Just before the maximum the slope is positive, at the maximum it is zero and just after the maximum it is negative. Thus, $\frac{dy}{dx}$ decreases at a maximum and hence the rate of change of $\frac{dy}{dx}$ is negative at a maximum i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right) < 0$ at maximum.

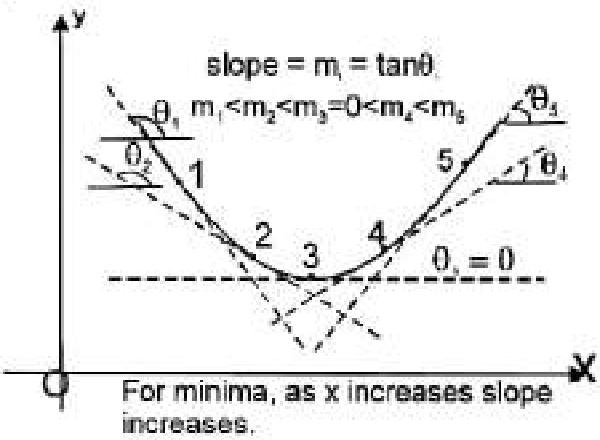



The quantity $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the rate of change of the slope. It is written as $\frac{d^2y}{dx^2}$.

**Conditions for maxima are:** (a) $\frac{dy}{dx} = 0$ (b) $\frac{d^2y}{dx^2} < 0$

**MINIMA**

Similarly, at a minimum the slope changes from negative to positive. Hence with the increases of x. the slope is increasing that means the rate of change of slope with respect to x is positive hence $\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$.




**Conditions for minima are :** (a) $\frac{dy}{dx} = 0$ (b) $\frac{d^2y}{dx^2} > 0$

Quite often it is known from the physical situation whether the quantity is a maximum or a minimum. The test on $\frac{d^2y}{dx^2}$ may then be omitted.

## *Solved Examples*

**Example 38.** Particle's position as a function of time is given as $x = 5t^2 - 9t + 3$. Find out the maximum value of position co-ordinate? Also, plot the graph.

**Solution :** $x = 5t^2 - 9t + 3$

$\frac{dx}{dt} = 10t - 9 = 0 \quad \therefore \quad t = 9/10 = 0.9$

Check, whether maxima or minima exists. $\frac{d^2x}{dt^2} = 10 > 0$

$\therefore$ there exists a minima at t = 0.9

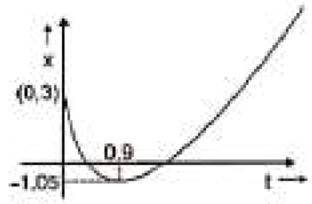
Now, Check for the limiting values.

When $t = 0$ ; $x = 3$

$t = \infty$ ; $x = \infty$

So, the maximum position co-ordinate does not exist.

**Graph :**




Putting $t = 0.9$ in the equation $x = 5(0.9)^2 - 9(0.9) + 3 = -1.05$

**NOTE :** If the coefficient of $t^2$ is positive, the curve will open upside.

## SOLVED EXAMPLES ON APPLICATION OF DERIVATIVE

**Example 39.** Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents ? If so, where ?

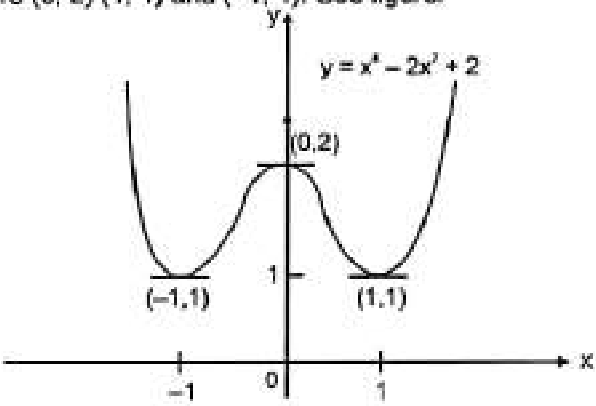
**Solution :** The horizontal tangents, if any, occur where the slope dy/dx is zero. To find these points. We

1. Calculate dy/dx : $\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x$

2. Solve the equation : $\frac{dy}{dx} = 0$ for x : $4x^3 - 4x = 0$

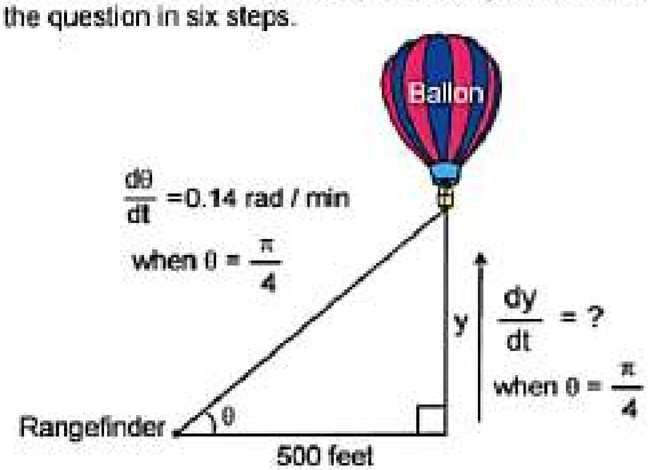
   $4x(x^2 - 1) = 0$ ; $x = 0, 1, -1$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0$, 1 and $-1$. The corresponding points on the curve are (0, 2) (1, 1) and (−1, 1). See figure.




**Example 40.** A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at the moment ?

**Solution :** We answer the question in six steps.



**Step 1 :** Draw a picture and name the variables and constants (Figure). The variables in the picture are

$\theta$ = the angle the range finder makes with the ground (radians).

y = the height of the balloon (feet).

We let t represent time and assume $\theta$ and y to be differentiable functions of t.

The one constant in the picture is the distance from the range finder to the lift-off point (500 ft.) There is no need to give it a special symbol s.

**Step 2 :** Write down the additional numerical information. $\frac{d\theta}{dt} = 0.14$ rad/min when $\theta = \pi / 4$.

**Step 3 :** Write down what we are asked to find. We want dy/dt when $\theta = \pi/4$.

**Step 4 :** Write an equation that relates the variables y and $\theta$. = tan $\theta$. $\frac{y}{500}$ or $y = 500 \tan\theta$.

**Step 5 :** Differentiate with respect to t using the Chain Rule. The result tells how dy/dt (which we want) is related to $d\theta/dt$ (which we know).

$$\frac{dy}{dt} = 500 \sec^2\theta \ \frac{d\theta}{dt}$$

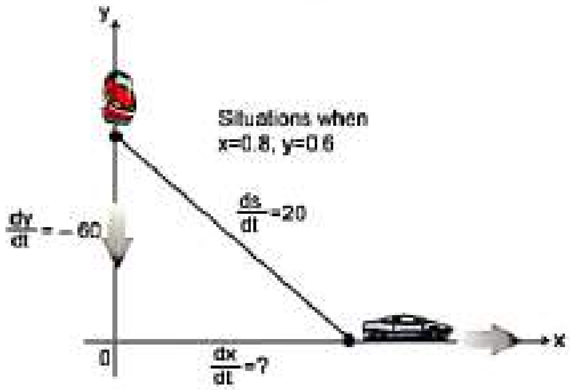
**Step 6 :** Evaluate with $\theta = \pi/4$ and $d\theta/dt = 0.14$ to find dy/dt.

$$\frac{dy}{dt} = 500\left(\sqrt{2}\right)^2 (0.14) = (1000)(0.14) = 140 \quad \left(\sec\frac{\pi}{4} = \sqrt{2}\right)$$

At the moment in question, the balloon is rising at the rate of 140 ft./min.

**Example 41.** A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the Cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

**Solution :** We carry out the steps of the basic strategy.




**Step 1 :** Picture and variables. We picture the car and cruiser in the coordinate plane, using the positive x-axis as the eastbound highway and the positive y-axis as the northbound highway (Figure).

We let t represent time and set x = position of car at time t.

y = position of cruiser at time t, s = distance between car and cruiser at time t.

We assume x, y and s to be differentiable functions of t.

$x = 0.8$ mi, $y = 0.6$ mi , $\frac{dy}{dt} = -60$ mph, $\frac{ds}{dt} = 20$ mph

(dy/dt is negative because y is decreasing.)

☞ **Step 2 :** To find : $\frac{dx}{dt}$

☞ **Step 3 :** How the variables are related : $s^2 = x^2 + y^2$ Pythagorean theorem

(The equation $s = \sqrt{x^2 + y^2}$ would also work.)

☞ **Step 4 :** Differentiate with respect to t. $2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$ Chain Rule

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{1}{\sqrt{x^2+y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

☞ **Step 5 :** Evaluate, with x = 0.8, y = 0.6, dy/dt = – 60, ds/dt = 20, and solve for dx/dt.

$$20 = \frac{1}{\underbrace{\sqrt{(0.8)^2 + (0.6)^2}}_{1}}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right) \Rightarrow 20 = 0.8\frac{dx}{dt} - 36 \Rightarrow \frac{dx}{dt} = \frac{20+36}{0.8} = 70$$

At the moment in question, the car's speed is 70 mph.

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# 4. INTEGRATION

In mathematics, for each mathematical operation, there has been defined an inverse operation.
For example - Inverse operation of addition is subtruction, inverse operation of multiplication is division and inverse operation of square is square root. Similarly there is a inverse operation for differentiation which is known as integration

## 4.1 ANTIDERIVATIVES OR INDEFINITE INTEGRALS

**Definitions :**

A function F(x) is an antiderivative of a function f(x) if $F'(x) = f(x)$ for all x in the domain of f.
The set of all antiderivatives of f is the indefinite integral of f with respect to x, denoted by

The function is the integrand.

x is the variable of integration

Integral sign

$$\underbrace{\int f(x)dx}_{\text{Integral of f}}$$

The symbol $\int$ is an integral sign. The function f is the integrand of the integral and x is the variable of integration.

For example $f(x) = x^3$ then $f'(x) = 3x^2$
So the integral of $3x^2$ is $x^3$
Similarly if $f(x) = x^3 + 4$ then $f'(x) = 3x^2$
So the integral of $3x^2$ is $x^3 + 4$
there for general integral of $3x^2$ is $x^3 + c$ where c is a constant

One antiderivative F of a function f, the other antiderivatives of f differ from F by a constant. We indicate this in integral notation in the following way :

$\int f(x)dx = F(x) + C.$ ......(i)

The constant C is the constant of integration or arbitrary constant, Equation (1) is read, "The indefinite integral of f with respect to x is F(x) + C." When we find F(x)+ C, we say that we have integrated f and evaluated the integral.

## Solved Examples

**Example 42.** Evaluate $\int 2x\,dx$.

**Solution :** $\int 2x\,dx = x^2 + C$

an antiderivative of 2x

the arbitrary constant

The formula $x^2 + C$ generates all the antiderivatives of the function 2x. The function $x^2 + 1$, $x^2 - \pi$, and $x^2 + \sqrt{2}$ are all antiderivatives of the function 2x, as you can check by differentiation. Many of the indefinite integrals needed in scientific work are found by reversing derivative formulas.

## 4.2 INTEGRAL FORMULAS

| **Indefinite Integral** | **Reversed derivative formula** |
|---|---|
| 1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$, n rational | $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$ |
| $\int dx = \int 1dx = x + C$ (special case) | $\frac{d}{dx}(x) = 1$ |
| 2. $\int \sin(Ax+B)dx = \frac{-\cos(Ax+B)}{A} + C$ | $\frac{d}{dx}\left(-\frac{\cos\ kx}{k}\right) = \sin kx$ |
| 3. $\int \cos kx dx = \frac{\sin\ kx}{k} + C$ | $\frac{d}{dx}\left(\frac{\sin kx}{k}\right) = \cos kx$ |
| 4. $\int \sec^2 x dx = \tan x + C$ | $\frac{d}{dx}\tan x = \sec^2 x$ |
| 5. $\int \operatorname{cosec}^2 x dx = -\cot x + C$ | $\frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x$ |
| 6. $\int \sec x \tan x\,dx = \sec x + C$ | $\frac{d}{dx}\sec x = \sec x \tan x$ |
| 7. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$ | $\frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x$ |
| 8. $\int \frac{1}{(ax+b)} = \frac{1}{a}\ell n(ax+b) + C$ | |

## Solved Examples

**Example 43.** Examples based on above formulas :

(a) $\int x^5 dx = \frac{x^6}{6} + C$ — Formula 1 with n = 5

(b) $\int \frac{1}{\sqrt{x}}dx = \int x^{-1/2}dx = 2x^{1/2} + C = 2\sqrt{x} + C$ — Formula 1 with n = –1/2

(c) $\int \sin 2x\,dx = \frac{-\cos 2x}{2} + C$ — Formula 2 with k = 2

(d) $\int \cos\frac{x}{2}dx = \int \cos\frac{1}{2}x dx = \frac{\sin(1/2)x}{1/2} + C = 2\sin\frac{x}{2} + C$ — Formula 3 with k = 1/2

**Example 44.** Right : $\int x \cos x \, dx = x \sin x + \cos x + C$

Reason : The derivative of the right-hand side is the integrand:

Check : $\frac{d}{dx}(x \sin x + \cos x + C) = x \cos x + \sin x - \sin x + 0 = x \cos x.$

Wrong : $x \cos x \, dx = x \sin x + C$

Reason : $\int$ The derivative of the right-hand side is not the integrand:

Check : $\frac{d}{dx}(x \sin x + C) = x \cos x + \sin x + 0 \quad x \cos x.$

## 4.3 RULES FOR INTEGRATION

### RULE NO. 1 : CONSTANT MULTIPLE RULE

A function is an antiderivative of a constant multiple kf of a function f if and only if it is k times an antiderivative of f.

$\int kf(x)dx = k\int f(x)dx$ ; where k is a constant

**Example 45.** Rewriting the constant of integration $\int 5 \sec x \tan x \, dx = 5 \int \sec x \tan x \, dx$

Rule 1

$= 5 (\sec x + C)$ Formula 6

$= 5 \sec x + 5C$ First form

$= 5 \sec x + C'$ Shorter form, where C' is 5C

$= 5 \sec x + C$ Usual form–no prime. Since 5 times an arbitrary constant is an arbitrary constant, we rename C'.

What about all the different forms in example? Each one gives all the antiderivatives of $f(x) = 5 \sec x \tan x$. so each answer is correct. But the least complicated of the three, and the usual choice, is

$$\int 5 \sec x \tan x \, dx = 5 \sec x + C.$$

Just as the Sum and Difference Rule for differentiation enables us to differentiate expressions term by term, the Sum and Difference Rule for integration enables us to integrate expressions term by term. When we do so, we combine the individual constants of integration into a single arbitrary constant at the end.

### RULE NO. 2 : SUM AND DIFFERENCE RULE

A function is an antiderivative of a sum or difference $f \pm g$ if and only if it is the sum or difference of an antiderivative of f an antiderivative of g.

$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

## Solved Examples

**Example 46.** Term–by–term integration. Evaluate : $\int (x^2 - 2x + 5)\,dx$.

**Solution :** If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5)dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + \underset{\text{arbitrary constant}}{C}$$

If we do not recognize the antiderivative right away, we can generate it term by term with the sum and difference Rule:

$$\int (x^2 - 2x + 5)dx = \int x^2 dx - \int 2x dx + \int 5 dx = \frac{x^3}{3} + C_1 - x^2 + C_2 + 5x + C_3.$$

This formula is more complicated than it needs to be. If we combine $C_1, C_2$ and $C_3$ into a single constant $C = C_1 + C_2 + C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

and still gives all the antiderivatives there are. For this reason we recommend that you go right to the final form even if you elect to integrate term by term. Write

$$\int (x^2 - 2x + 5)dx = \int x^2 dx - \int 2x dx + \int 5 dx = \frac{x^3}{3} - x^2 + 5x + C.$$

Find the simplest antiderivative you can for each part add the constant at the end.

**Example 47.** We can sometimes use trigonometric identities to transform integrals we do not know how to evaluate into integrals we do know how to evaluate. The integral formulas for $\sin^2 x$ and $\cos^2 x$ arise frequently in applications.

(a) $\int \sin^2 x dx = \int \frac{1 - \cos 2x}{2} dx \quad \sin^2 x = \frac{1 - \cos 2x}{2}$

$$= \frac{1}{2}\int (1 - \cos 2x)dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{x}{2} - \left(\frac{1}{2}\right)\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) $\int \cos^2 x dx = \int \frac{1 + \cos 2x}{2} dx \quad \cos^2 x = \frac{1 + \cos 2x}{2}$

$$= \frac{x}{2} + \frac{\sin 2x}{4} + C$$ As in part (a), but with a sign change

**Example 48.** Find a body velocity from its acceleration and initial velocity. The acceleration of gravity near the surface of the earth is 9.8 m/sec$^2$. This means that the velocity **v** of a body falling freely in a vacuum changes at the rate of $\frac{dv}{dt} = 9.8$ m/sec$^2$. If the body is dropped from rest, what will its velocity be **t** seconds after it is released?

**Solution :** In mathematical terms, we want to solve the initial value problem that consists of

The differential condition : $\frac{dv}{dt} = 9.8$

The initial condition : $v = 0$ when $t = 0$ ( abbreviated as $v(0) = 0$)

We first solve the differential equation by integrating both sides with respect to t:

| | |
|---|---|
| $\frac{dv}{dt} = 9.8$ | The differential equation |
| $\int \frac{dv}{dt} dt = \int 9.8 dt$ | Integrate with respect to t. |
| $v + C_1 = 9.8t + C_2$ | Integrals evaluated |
| $v = 9.8t + C.$ | Constants combined as one |

This last equation tells us that the body's velocity t seconds into the fall is $9.8t + C$ m/sec.

For value of C : What value? We find out from the initial condition :

$v = 9.8t + C$

$0 = 9.8(0) + C$ $\quad$ $v(0) = 0$

$C = 0.$

Conclusion : The body's velocity t seconds into the fall is

$v = 9.8t + 0 = 9.8t$ m/sec.

The indefinite integral $F(x) + C$ of the function $f(x)$ gives the general solution $y = F(x) + C$ of the differential equation $dy/dx = f(x)$. The general solution gives all the solutions of the equation (there are infinitely many, one for each value of C). We solve the differential equation by finding its general Solution : We then solve the initial value problem by finding the particular solution that satisfies the initial condition $y(x_0) = y_0$ ( y has the value $y_0$ when $x = x_0$ ).

**RULE NO. 3 : RULE OF SUBSTITUTION**

$\int f(g(x)) \cdot g'(x)\, dx = \int f(u) du$ 1. Substitute $u = g(x)$, $du = g'(x)\, dx$.

$= F(u) + C$ 2. Evaluate by finding an antiderivative F (u) of f(u). (any one will do.)

$= F(g(x)) + C$ 3. Replace u by g(x).

## *Solved Examples*

**Example 49.** Evaluate $\int (x+2)^5 dx$.

We can put the integral in the form $\int u^n du$

by substituting $u = x + 2$, $du = d(x + 2) = \frac{d}{dx}(x + 2).dx = 1.dx = dx.$

Then $\int (x+2)^5 dx = \int u^5 du$ $u = x + 2$, $du = dx$

$= \frac{u^6}{6} + C$ Integrate, using rule no. 3 with n= 5.

$= \frac{(x+2)^6}{6} + C.$ Replace u by x + 2.

**Example 50.** Evaluate $\int\sqrt{1+y^2}\cdot 2y\,dy = \int u^{1/2}du$. Let $u = 1 + y^2$, $du = 2y\,dy$.

$= \dfrac{u^{(1/2)+1}}{(1/2)+1}$ Integrate, using rule no. 3 with n = 1/2.

$= \dfrac{2}{3}u^{3/2} + C$ Simpler form

$= \dfrac{2}{3}(1+y^2)^{3/2} + C$ Replace u by $1 + y^2$.

**Example 51.** Evaluate $\int\cos\,(7\theta + 5)\,d\theta = \int\cos u\cdot\dfrac{1}{7}du$ Let $u = 7\theta + 5$, $du = 7d\theta$, $(1/7)\,du = d\theta$.

$= \dfrac{1}{7}\int\cos u\,du$ With (1/7) out front, the integral is now in standard form.

$= \dfrac{1}{7}\sin u + C$ Integrate with respect to u.

$= \dfrac{1}{7}\sin\,(7\theta + 5) + C$ Replace u by $7\theta + 5$.

**Example 52.** Evaluate $\int x^2\sin(x)^3dx = \int\sin(x)^3\cdot x^2\,dx$

$= \int\sin u\cdot\dfrac{1}{3}du$ Let $u = x^3$, $du = 3x^2\,dx$, $(1/3)\,du = x^2dx$.

$= \dfrac{1}{3}\int\sin u\,du$

$= \dfrac{1}{3}\,(-\cos u) + C$ Integrate with respect to u.

$= -\dfrac{1}{3}\cos(x^3) + C$ Replace u by $x^3$.

**Example 53.** $\int\dfrac{1}{\cos^2 2\theta}d\theta = \int\sec^2\ 2\theta\ d\theta\sec 2\theta = \dfrac{1}{\cos 2\theta}$

$= \int\sec^2u\cdot\dfrac{1}{2}du$ Let $u = 2\theta$, $du = 2d\theta$, $d\theta = (1/2)du$.

$= \dfrac{1}{2}\int\sec^2 u\,du$

$= \dfrac{1}{2}\tan u + C$ Integrate, using eq. (4).

$= \dfrac{1}{2}\tan 2\theta + C$ Replace u by $2\theta$.

**Check:** $\dfrac{d}{d\theta}\left(\dfrac{1}{2}\tan 2\theta + C\right) = \dfrac{1}{2}\cdot\dfrac{d}{d\theta}(\tan 2\theta) + 0 = \dfrac{1}{2}\cdot\left(\sec^2 2\theta\cdot\dfrac{d}{d\theta}2\theta\right)$ Chain Rule

$= \dfrac{1}{2}\cdot\sec^2 2\theta\cdot 2 = \dfrac{1}{\cos^2 2\theta}$.

**Example 54.** $\int \sin^4 t \cos t\,dt = \int u^4 du$ Let $u = \sin t$, $du = \cos t\,dt$.

$= \frac{u^5}{5} + C$ Integrate with respect to u.

$= \frac{\sin^5 t}{5} + C$ Replace u.

The success of the substitution method depends on finding a substitution that will change an integral we cannot evaluate directly into one that we can. If the first substitution fails, we can try to simplify the integrand further with an additional substitution or two.

## 4.3 DEFINITE INTEGRATION OR INTEGRATION WITH LIMITS

The function is the Integrand.

Upper limit of integration

x is the variable of integration

Integral sign

$$\underbrace{\int_a^b f(x)dx}_{\text{Integral of f from a to b}}$$

Lower limit of integration

$$\int_a^b f(x)dx = [g(x)]_a^b = g(b) - g(a)$$

where g(x) is the antiderivative of f(x) i.e. $g'(x) = f(x)$

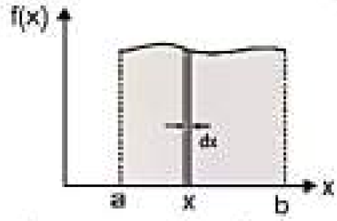
### *Solved Examples*

**Example 55.** $\int_{-1}^{4} 3dx = 3\int_{-1}^{4} dx = 3[x]_{-1}^{4} = 3[4-(-1)] = (3)(5) = 15$

$$\int_0^{\pi/2} \sin x dx = [-\cos x]_0^{\pi/2} = -\cos\left(\frac{\pi}{2}\right) + \cos(0) = -0 + 1 = 1$$

## 4.4 APPLICATION OF DEFINITE INTEGRAL : CALCULATION OF AREA OF A CURVE

From graph shown in figure if we divide whole area in infinitely small strips of dx width. We take a strip at x position of dx width. Small area of this strip $dA = f(x)\,dx$




So, the total area between the curve and x-axis

$= \text{sum of area of all strips} = \int_a^b f(x)dx$

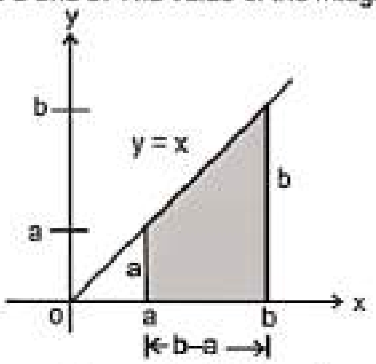
Let $f(x) \geq 0$ be continuous on [a, b]. The area of the region between the graph of f and the x-axis is

$$A = \int_a^b f(x)dx$$

## Solved Examples

**Example 56.** Using an area to evaluate a definite integral $\int_a^b x dx$ $0 < a < b$.

**Solution :** We sketch the region under the curve $y = x$, $a \le x \le b$ (figure) and see that it is a trapezoid with height $(b - a)$ and bases a and b. The value of the integral is the area of this trapezoid :




The region in Example

$$\int_a^b x\, dx = (b - a) \cdot \frac{a + b}{2} = \frac{b^2}{2} - \frac{a^2}{2}.$$

Notice that $x^2/2$ is an antiderivative of x, further evidence of a connection between antiderivatives and summation.