:- LOGARITHM :-

Introduction :-

When we are given the base 2, and exponent 3, we can calculate $2^3 = 8$. Inversely, if we are given base 2 and its value 8, and we need to find the exponent, $2^{\cdots} = 8$.

This exponent is called 'Logarithm'. we can call the exponent 3, as the "logarithm" of 8 with base 2" and we write it as $3 = \log_2 8$.

Thus if $a^x = b$, this can be written as '$\log_a b = x$'

In general, the logarithm of a positive number 'x' with respect to a positive base number $a (\neq 1)$ is the exponent (or, index) of the power of 'a' to which 'a' is to be raised to attain the number 'x'. This index of the power of 'a' is written as, '$\log_a x$' & read as logarithm or, log of 'x' to the base 'a'.

Note:- only positive real nos $(m > 0)$ in $\log_a m$ have real logarithms; negative & complex numbers have complex logarithms.

Now, Lets take $a = -3$

$$\therefore (-3)^1 = -3 \Rightarrow \log_{-3} -3 = 1$$

$$\therefore (-3)^2 = 9 \Rightarrow \log_{-3} 9 = 2$$

$$\therefore (-3)^3 = -27 \Rightarrow \log_{-3} -27 = 3$$

Hence, we get that the value of 'log' increases for irregular nos. Thus, it implies that in,

$$\log_a b = x, \quad a > 0$$

Now, for $a = 1$,

$$1^1 = 1 \Rightarrow \log_1 1 = 1$$

$$1^2 = 1 \Rightarrow \log_1 1 = 2$$

$$1^3 = 1 \Rightarrow \log_1 1 = 3$$

Here, also without increase in base, or, value, the 'log' increases which is irregularity, so, base should not be equal to 1.

Hence, in $\log_a b = x$, $a > 0$ & $a \neq 1$

* For regularity 'log' need not increase, it may decrease uniformly,

Note:- when base 'e' then the logarithmic funcⁿ is called 'natural logarithm' & when base is 10, then it is called 'common logarithm'.

$$e = 2.71828$$

$$\log_{10} e = 0.4342 \ \& \ \log_e 10 = \frac{1}{\log_{10} e} = 2.302.$$

Also, $\boxed{\log_e x = \ln x}$

Special logarithms :-

we have $a^0 = 1$, where $a > 0$ & $a \neq 1$. so $\log_a 1 = 0$

Also, $a^1 = a$, so, $\log_a a = 1$.

Thus, we have two special logarithms,

$$\log_a 1 = 0 \ \& \ \log_a a = 1.$$

Properties :-

(1) $a^{\log_a m} = m$ (fundamental logarithms identity).

Proof:- from defⁿ $\log_a m = x$ ---(i) $\Rightarrow a^x = m$ ---(ii)

putting the value of 'x' in (ii)

i.e. $\boxed{a^{\log_a m} = m}$

(2) $\log_a x^n = n \log_a x$

Proof:- $\log_a x^n = t \Rightarrow x^n = a^t$

$\therefore \ x = a^{t/n}$

i.e. $\log_a x = t/n \Rightarrow n \log_a x = t$

so, $\boxed{\log_a x^n = n \log_a x}$

(3) $\log_{a^m}(x) = \frac{1}{m} \log_a x$

Proof: $\log_{a^m} x = t$ (say)

$\Rightarrow x = (a^m)^t = a^{mt}$

i.e. $\log_a x = mt \Rightarrow t = \frac{1}{m} \log_a x$

so, $\boxed{\log_{a^m} x = \frac{1}{m} \log_a x}$

(4) $\log_c a = \dfrac{\log_b a}{\log_b c} = \log_b a \times \log_c b$ (Base change property).

Proof:- Let $\log_b a = x \Rightarrow a = b^x$,

$\log_c a = y \Rightarrow a = c^y$

& $\log_b c = z \Rightarrow c = b^z$

Hence, $a = c^y = (b^z)^y = b^{yz}$

Also, $a = b^x = b^{yz} \Rightarrow x = yz$

$\therefore \log_b a = \log_c a \; \log_b c$

In general,

$$\boxed{\log_b a = \log_c a \times \log_d c \times \log_e d \times \cdots \log_b k}$$

Also, $\log_c a = \dfrac{\log_b a}{\log_b c}$

Simillarly, $\boxed{\log_a a = \dfrac{\log_b a}{\log_b a} = 1 \quad \& \quad \log_b a = \dfrac{1}{\log_a b}}$

(5) $\log_a (m\cdot n) = \log_a m + \log_a n$.

Proof: Let, $\log_a m = x$ & $\log_a n = y$

$\Rightarrow a^x = m$ --- (i) & $a^y = n$ --- (ii)

Multiplying (i) & (ii),

$a^{x+y} = m \times n \Rightarrow \log_a (m\times n) = x+y$

i.e. $\boxed{\log_a (m\times n) = \log_a m + \log_a n}$

(6) $\log_a (m/n) = \log_a m - \log_a n$

Proof:- Let, $\log_a m = x$ & $\log_a n = y \Rightarrow a^x = m$ & $a^y = n$

So, $\dfrac{a^x}{a^y} = m/n \Rightarrow a^{x-y} = m/n \Rightarrow \log_a (m/n) = x - y$

i.e. $\boxed{\log_a (m/n) = \log_a m - \log_a n}$

(7) $\log_a (1/x) = -\log_a x$

(8) $a^{\log_b c} = c^{\log_b a}$

Proof:- Let $k = a^{\log_b c} \Rightarrow \log k = \log(a^{\log_b c}) = \log_b c \,(\log a)$

i.e. $\log k = \dfrac{\log c}{\log b} \cdot \log a = \log_b a \,(\log c) = \log(c^{\log_b a})$

$\therefore k = e^{\log_b a}$

Q.1. Express each in 'log' form:

(a) $4^3 = 64$ (b) $5^x = 20$ (c) $(1/3)^4 = 1/81$

Soln:- (a) As, we know, $a^x = b \Rightarrow x = \log_a b$

So, $4^3 = 64 \Rightarrow 3 = \log_4 64$

Students can try rest questions.

Q.2. Find x,

(a) $\log_{10} x = -3$ (b) If $\log_{10} x = a$, express 10^{2a-3} in terms of x.

(c) If $\log_{10} x = a$, $\log_{10} y = b$ & $\log_{10} z = 2a - 3b$ then express 'z' in terms of x & y.

Soln:-

(a) $\log_{10} x = -3$

So, by the defn of logarithm,

$$x = 10^{-3} \Rightarrow x = \frac{1}{1000}$$

(b) Here, $a = \log_{10} x$

$$\therefore 10^{2a-3} = 10^{2\log_{10} x - 3} = 10^{\log_{10} x^2 - 3} = (10)^{\log_{10}(x^2)} - 3$$

$$= x^2 \times (10)^{-3} = \frac{x^2}{1000}.$$

(c) $a = \log_{10} x$, $b = \log_{10} y$

Here, $\log_{10} z = 2a - 3b \Rightarrow z = 10^{2a-3b}$

Now, $2a - 3b = 2\log_{10} x - 3\log_{10} y = \log_{10} x^2 - \log_{10} y^3 = \log_{10}\left(\frac{x^2}{y^3}\right)$

$$\therefore z = 10^{\log_{10}(x^2/y^3)}$$

So, $\boxed{z = x^2/y^3}$

Q.3. Students must try,

(a) Evaluate: $\log_5\left(\frac{1}{125}\right)$ (b) $\frac{\log 8}{\log \sqrt{2}}$

Q.4. If $x = \log(3/5)$, $y = \log(5/4)$ & $z = 2\log(\sqrt{3}/2)$, then prove that $5^{x+y-z} = 1$.

Sol^n:- Here, $x = \log(3/5) = \log 3 - \log 5$, $y = \log(5/4) = \log 5 - \log 4$.

& $z = 2\log(\sqrt{3}/2) = \log(3/4) = \log 3 - \log 4$

$\therefore\ 5^{x+y-z} = 5^{\cancel{\log 3} - \cancel{\log 5} + \cancel{\log 5} - \cancel{\log 4} - \cancel{\log 3} + \cancel{\log 4}} = 5^0 = 1.$

Q.5. Evaluate: $2\log 3 - \frac{1}{2}\log 16 + \log 12$.

Sol^n:- Here, $2\log 3 - \log 16^{1/2} + \log(3\times 4)$

$= \log 3^2 - \cancel{\log 4} + \log 3 + \cancel{\log 4}$

$= \log 9 + \log 3$

$= \log(27)$

$= \log 3^3$

$= 3\log 3.$

Q.6. Prove that: $9^{\log 4} = 16^{\log 3}$.

Sol^n:- Here, LHS $= 9^{\log 4} = 4^{\log 9} = 4^{\log 3^2} = 4^{2\log 3} = 16^{\log 3}$ = RHS.

Q7. Solve: $\log_{x_1} x_2 \times \log_{x_2} x_3 \times \log_{x_3} x_4 \times \cdots \times \log_{x_{n-1}} x_n \times \log_{x_n} x_1$.

Sol^n:- use Base change property & try this.

Q.8. $4^{\log_9 3} + 9^{\log_2 4} = 10^{\log_x 83}$ then find 'x'.

Sol^n:- Here, $4^{\log_{3^2} 3} + 9^{2\log_2 2} = 10^{\log_x 83} \Rightarrow 4^{\frac{1}{2}\log_3 3} + 9^2 = 10^{\log_x 83}$

or $2 + 81 = 10^{\log_x 83} \Rightarrow 83 = 83^{\log_x 10} \Rightarrow 1 = \log_x 10$

$\therefore\ x = 10.$

Q.9. If $x > 1$, find the min^m value of $2\log_{10} x - \log_x 0.01$.

→ Students must try this.