(29) $f(x) = x^3 - 3x + 1$, $g(x) = 3x^2$, then if α, β, γ be the roots of $f(x) = 0$,

(a) find $\alpha^5 + \beta^5 + \gamma^5$

(b) $[\alpha] + [\beta] + [\gamma]$

(c) find the no. of real roots of $f(f(x)) = 0$.

(d) If α, β, γ be the roots of $f(x) = g(x)$. Then PT: $\alpha+\beta, \beta+\gamma, \gamma+\alpha$ be the roots of $f(3-x) - g(x-3) = 0$.

Sol^n:-

Its not easy to find roots of $f(x)=0$, then we know α, β & γ be the roots of $f(x) = 0$.

i.e. $\alpha^3 - 3\alpha + 1 = 0 \Rightarrow \alpha^3 = 3\alpha - 1$.

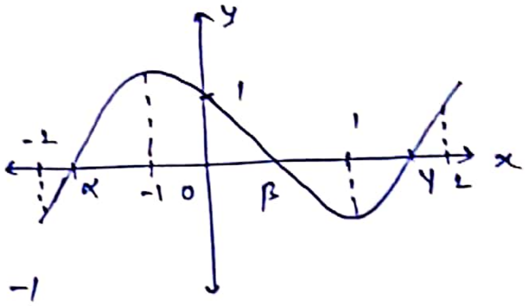
(a) $\therefore \alpha^5 = \alpha^3 \cdot \alpha^2 = (3\alpha - 1)\alpha^2 = 3\alpha^3 - \alpha^2 = 3(3\alpha - 1) - \alpha^2$
$= 9\alpha - 3 - \alpha^2$

so, $\alpha^5 + \beta^5 + \gamma^5 = 9(\alpha+\beta+\gamma) - 3\times 3 - (\alpha^2+\beta^2+\gamma^2) = 0 \times 9 - 9 - 6$
$= \boxed{-15}$

Students must try: $\alpha^7 + \beta^7 + \gamma^7 = ?$

(b) purpose, is not to find the roots exactly, we just have to find location, $\therefore f'(x) = 3(x^2 - 1) = 0 \Rightarrow x = \pm 1$.

Lesser one is local max^m. Since its cubic polynomial & hence, it starts with -ve' infinity in the graph, if coeff. is positive.



$f(2) > 0$, Then $1 < \gamma < 2$

$f(-2) < 0$, then $-2 < \alpha < -1$

Also, $0 < \beta < 1$

$\therefore [\alpha] + [\beta] + [\gamma] = -2 + 0 + 1 = -1$

so, $\boxed{[\alpha] + [\beta] + [\gamma] = -1}$

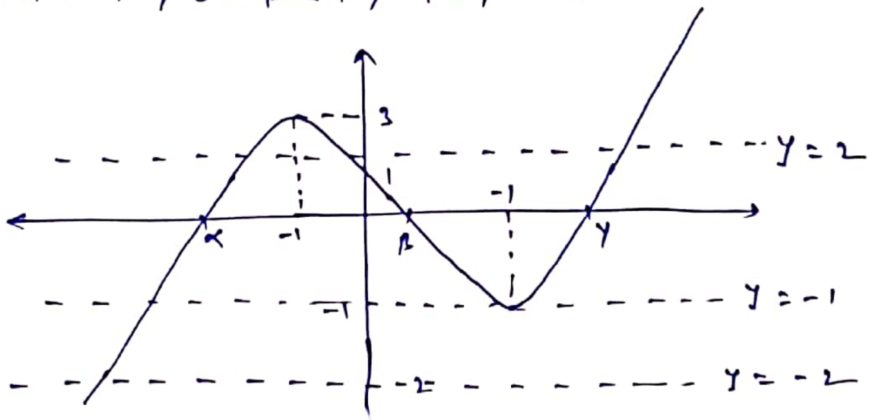
(c) If $f(x) = 0 \Rightarrow x = \alpha, \beta, \gamma$

$f(f(x)) = 0 \Rightarrow f(x) = \alpha, \beta, \gamma$.

To find number of sol^n for $f(x) = g(x)$, then number of times their graphs intersect

using the above mentioned method, we make the graph of $f(x)$ & $f(x)$

Now, $-2 < \alpha < -1$, $0 < \beta < 1$, $1 < \gamma < 2$.



$y = \alpha$ intersect curve once, $y = \beta$ intersect curve thrice, & $y = \gamma$ intersect curve thrice.

$\therefore$ $\boxed{f(f(x)) \text{ has '7' real roots.}}$

(d.) Here, $f(x) = g(x) \Rightarrow x^3 - 3x + 1 = 3x^2$

$\Rightarrow x^3 - 3x^2 - 3x + 1 = 0 \rightarrow \alpha, \beta, \gamma$

we have to find eq. with roots $\alpha+\beta$, $\beta+\gamma$ & $\gamma+\alpha$.

$\therefore \alpha + \beta + \gamma = 3$

$\Rightarrow \alpha + \beta = 3 - \gamma$, $\beta + \gamma = 3 - \alpha$, $\gamma + \alpha = 3 - \beta$.

symmetrical situation, so, putting $x = 3 - \alpha \Rightarrow \alpha = 3 - x$ in $f(x) - g(x) = 0$, we get, eqn. with roots $\alpha+\beta$, $\beta+\gamma$, $\alpha+\gamma$.

Now, $f(\alpha) = g(\alpha) \Rightarrow f(3-x) = g(3-x)$

$\Rightarrow f(3-x) = g(3-x)$

$\Rightarrow f(3-x) - g(3-x) = 0$

$\Rightarrow f(3-x) - g(x-3) = 0$ (as $g(-x) = g(x) \rightarrow$ even fn).

Hence, $\alpha+\beta$, $\beta+\gamma$, $\alpha+\gamma$ are the roots.

(20.) If $x^4 - 8x^3 + 22x^2 - 24x + p = 0$ be the polynomial, then find set of values of 'p' such that

(a) It has all roots complex.

(b) It has all roots real.

(c) Both real & complex roots.

sol^n:- Here, $f'(x) = 4(x^3 - 6x^2 + 11x - 6) = 0 \Rightarrow x = 1, 2, 3.$

Since, the polynomial of even degree, starts with positive infinity & ends at '+ve' infinity if co-eff. of x^n is '+ve'. Now, the middle term must be local max^m. since the co-eff. of x^4 is positive here.

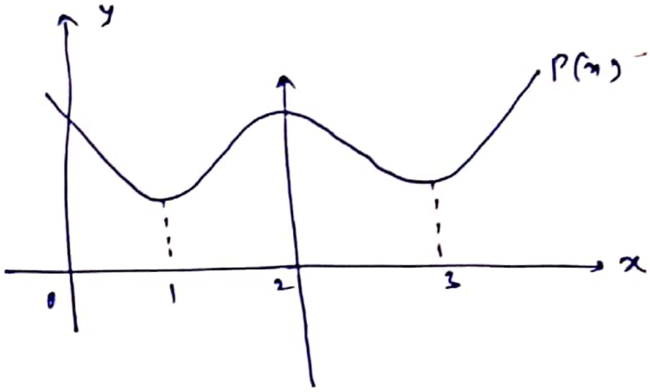
(a)

So, for roots to be complex.

$P(1) > 0$ & $P(3) > 0$

i.e. $p - 9 > 0$ & $p - 9 > 0$

so, $\boxed{p > 9}$ .... (1)



(b)

from both the cases,

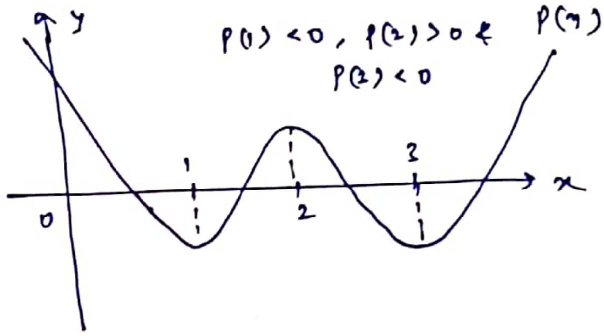
$P(1) \le 0$, $P(2) \geq 0$ & $P(3) \le 0$

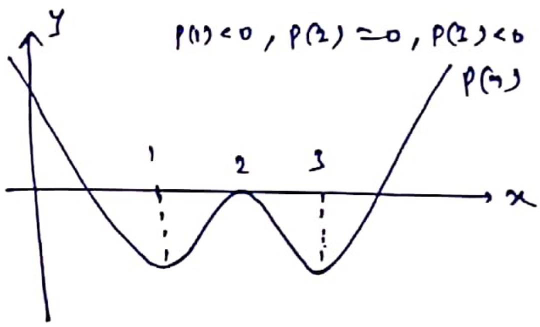
$\therefore$ $f(1) = p - 9 \le 0 \Rightarrow p \le 9$

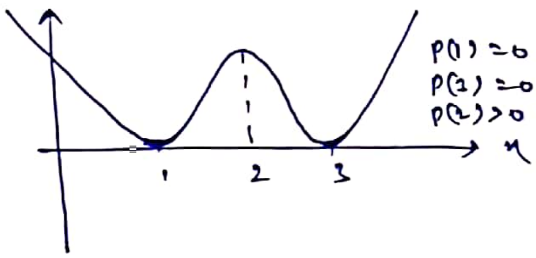
$f(3) = p - 9 \le 0 \Rightarrow p \le 9$

$f(2) \geq 0 \Rightarrow p \geq 8$

so, $\boxed{p \in [8, 9]}$





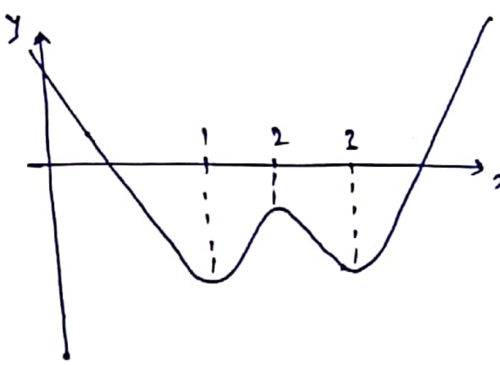


(c) For this case;

$f(2) < 0 \Rightarrow p - 8 < 0$

so, $p < 8$

i.e. $\boxed{p \in (-\infty, 8)}$



(31.) Let $f(x) = x^3 + x^2 + 100x + 7\sin x$, then eqⁿ:

$$\frac{1}{y - f(1)} + \frac{2}{y - f(2)} + \frac{3}{y - f(3)} = 0 \text{ has}$$

(a) no real roots (b) one real root $\cancel{(c)}$ two real roots (d) More than two real roots.

Solⁿ:- Here, $f'(x) = 3x^2 + 2x + 100 + 7\cos x$

$$= 3\left\{(x + \tfrac{1}{3})^2\right\} + \frac{278}{3} + 7(1 + \cos x) > 0$$

so, $f(x)$ is increasing fⁿ.

$$\therefore \underset{a}{f(1)} < \underset{b}{f(2)} < \underset{c}{f(3)} \Rightarrow a < b < c.$$

$$\therefore \frac{1}{y-a} + \frac{1}{y-b} + \frac{1}{y-c} = 0 \Rightarrow \text{let } g(y) = (y-b)(y-c) + 2(y-c)(y-a) + 2(y-a)(y-b)$$

$\therefore g(a) > 0,\ g(b) < 0\ \&\ g(c) > 0.$

Hence, one real root betⁿ a & b other betⁿ b & c

so, $\boxed{\text{option 'c'}}$

(32.) If α, β be the roots of $x^2 - x - 1 = 0$ & $A_n = \alpha^n + \beta^n$, then A.M. of A_{n-1} & A_n is ...

(a) $2A_{n+1}$ $\cancel{(B)}$ $\frac{1}{2}A_{n+1}$ (c) $2A_{n-2}$ (D) None.

Solⁿ:- Here, $\alpha + \beta = 1$ & $\alpha\beta = -1$

$$\therefore \frac{A_{n-1} + A_n}{2} = \frac{\alpha^{n-1} + \beta^{n-1} + \alpha^n + \beta^n}{2} = \frac{1}{2}(\alpha^{n+1} + \beta^{n+1})$$

$$= \frac{1}{2}A_{n+1}.$$

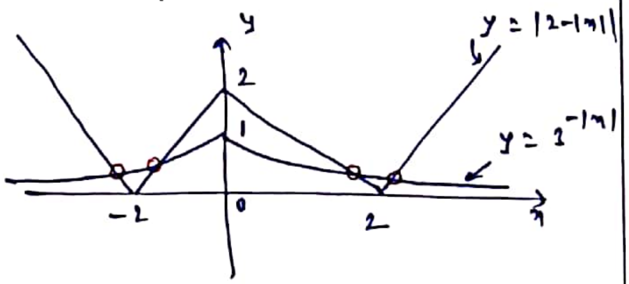
So, $\boxed{\text{option 'B'}}$

(33.) How many roots does the following eqⁿ possess.?

$$3^{|x|}\,|2 - |x|| = 1$$

(a) 1 (b) 2 (c) 3 $\cancel{(d)}$ 4.

Solⁿ:- Here, $|2 - |x|| = 3^{-|x|}$



So, clearly we can see there are four solutions

i.e. $\boxed{\text{option 'd'}}$

Quadratic expression :-

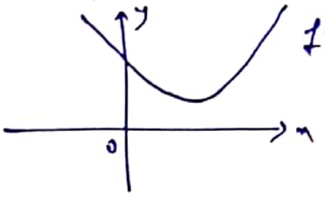
let $p(x) = ax^2+bx+c$.

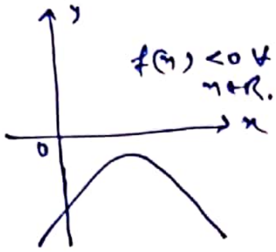
For $D<0$, suppose $\alpha = p+iq$ & $\beta = p-iq$ be the roots of $p(x)=0$.

Now, $p(x) = a(x-p-iq)(x-p+iq) = a\{(x-p)^2+q^2\}$.

so, clearly, $p(x)$ depends upon 'a'

i.e. If $a>0$, $f(x)>0 \; \forall \; x \in R$ & if $a<0$ $f(x)<0 \; \forall \; x \in R$.





In general, $\boxed{\begin{aligned} f(x) &= ax^2+bx+c>0 \;\forall\; x\in R \text{ if } D<0, a>0 \\ &= ax^2+bx+c<0 \;\forall\; x\in R \text{ if } D<0, a<0 \end{aligned}}$.

(32.) $f(x) = ax^2+bx+c=0$ has no real roots & $a+c>b$. Then which are correct

(a) $4a+2b+c>0$ (b) $4a-2b+c<0$ (c) $a-2b+4c<0$ (d) $a+2b+4c>0$

sol^n :- Here, $D<0$, so either it's always '+ve' or, negative.

Now, $f(-1) = a-b+c>0$ so, $f(x)>0 \; \forall \; x\in R$.

$f(2) = 4a+2b+c>0$, $f(-2) = 4a-2b+c>0$, $f(-\frac{1}{2}) = a-2b+4c>0$

& $f(\frac{1}{2}) = a+2b+4c>0$.

so, $\boxed{\text{options 'a, d'.}}$

(34.) If $a,b,c \in R^+$ such that $ax^2+bx+c \geq 0$, $bx^2+cx+a \geq 0$ & $cx^2+ax+b \geq 0 \; \forall \; x\in R$. Then pt: $1 < \frac{a^2+b^2+c^2}{ab+bc+ca} < 4$.

sol^n :- Here, $b^2 \leq 4ac$, $c^2 \leq 4ab$ & $a^2 \leq 4ac$.

Adding, $b^2+c^2+a^2 \leq 4(ab+bc+ca) \Rightarrow \frac{b^2+c^2+a^2}{bc+ca+ab} \leq 4$ ---(A)

Now, $\frac{a^2+b^2}{2} \geq (a^2 \times b^2)^{1/2} \longrightarrow (AM \geq GM)$

$\frac{b^2+c^2}{2} \geq (b^2\times c^2)^{1/2}$ & $\frac{a^2+c^2}{2} \geq (a^2 \times c^2)^{1/2}$

so, $\frac{a^2+b^2+c^2}{ab+bc+ca} \geq 1$ --- (B)

from A & B. $\boxed{1 \leq \frac{a^2+b^2+c^2}{ab+bc+ca} \leq 4}$

i.e. ~~option~~ Proved.