Another example :-

$$f(x) = x^3 - 1$$

Here, $+ \; -$ → one change in sign.

So, $f(x)$ has only one positive real root.

Also, $f(-x) = -x^3 - 1$

i.e. $- \; -$ → No change in sign.

So, $f(x)$ doesn't have any negative real roots.

∴ No. of imaginary roots = Total roots − Real roots
$= 3 - 1$
$= 2.$

By checking $x^3 - 1 = 0 \Rightarrow (x-1)(x^2+x+1) = 0$

i.e. $x = 1$

so, only one one real root & rest two are img.

24. Q. Find min^m. possible positive values of a, b, c, d if roots of $x^5 + ax^4 + bx^3 + cx^2 + dx + 1 = 0$ are all real, $(a, b, c, d > 0)$.

Sol^n :- Here, $p(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + 1$

so, by descarte's rule, $f(x)$ has no positive real root

But, it has 5 negative real roots.

Now, let x_1, x_2, x_3, x_4, x_5 be the roots of $p(x) = 0$

∴ $x_1 + x_2 + x_3 + x_4 + x_5 = -a$

∴ $x_1 x_2 + x_2 x_3 + \cdots = b$

∴ $x_1 x_2 x_3 + x_2 x_3 x_4 + \cdots = -c$

∴ $x_1 x_2 x_3 x_4 + x_2 x_3 x_4 x_5 + \cdots = d$

& $x_1 x_2 x_3 x_4 x_5 = -1.$

We know, AM ≥ GM.

so, $$\frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} \geq \{x_1 x_2 x_3 x_4 x_5\}^{1/5}$$

or, $\cancel{\frac{-a}{5}} \geq, = \dfrac{(-x_1)+(-x_2)+(-x_3)+(-x_4)+(-x_5)}{5} \geq \{(-x_1)(-x_2)(-x_3)(-x_4)(-x_5)\}^{1/5}$

or $a/5 \geq 1 \Rightarrow a \geq 5$

so, $\boxed{a_{min} = 5}$

Similarly, $\boxed{b_{min} = 10}$, $\boxed{c_{min} = 10}$, $\boxed{d_{min} = 5}$

Note :-

consider, $p(x) = (x-\alpha)^r \phi(x)$, where '$\alpha$' is r times repeated roots of $p(x) = 0$ & the $\phi(x)$ having degree '$n-r$'.

$$\therefore p'(x) = r(x-\alpha)^{r-1}\phi(x) + (x-\alpha)^r \phi'(x)$$

$$= (x-\alpha)^{r-1}\{r\phi(x) + (x-\alpha)\phi'(x)\}$$

$$= (x-\alpha)^{r-1}\phi_1(x)$$

Hence, 'α' is also a root of $p'(x) = 0$, which is $(r-1)$ times repeated.

$$\therefore p''(x) = (x-\alpha)^{r-2}\phi_2(x)$$

i.e. 'α' be the root of $p''(x) = 0$, which is $(r-2)$ times repeated.

$\therefore p^{r-1}(x) = (x-\alpha)\phi_{r-1}(x) \longrightarrow$ 'α' be the root of $p^{r-1}(x) = 0$.

Also, $p^r(x) = 0$ having no roots as 'α'.

Now, $p(x) = (x-\beta)^r \phi(x)$

$$\therefore f(x) = \int p(x)\,dx = (x-\beta)^{r+1}\psi(x) + c$$

if $c = 0$, then 'α' is a root otherwise not

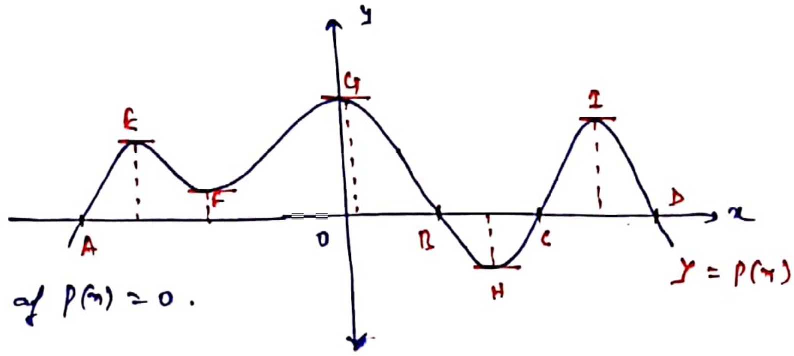
Students must Remember :→

* $p(x) = 0$ has 'k' ~~times~~ real roots then $p'(x) = 0$ as atleast $(k-1)$ real roots. (can be even more).

* If $p'(x) = 0$ has n real roots, $p(x) = 0$ can't have more than $(n+1)$ real roots.

we can see from the graphical Representation below :.

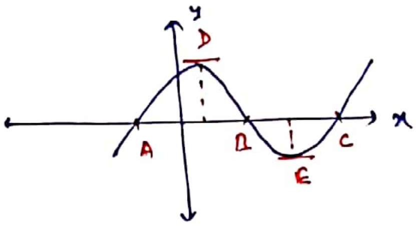
For the 1st case,



∴ A, B, C, D → roots of $P(x) = 0$.

& E, F, G, H, I → roots of $P'(x) = 0$.

For this case,



at A, B, C → $f(x) = 0$

& at D & E → $f'(x) = 0$.

Also, this is the minm. case.

25. Q. P(x) is the polynomial of degree 5. If P(x) is divided by $(x-1)^2$ remainder is -1. If P(x) is divided by $(x+1)^2$, remainder is 1.

(a). Find sum of roots of P(x)

(b). Find product of roots of P(x)

Soln :- According to question,

$P(x) = (x-1)^2 Q(x) - 1 \Rightarrow P(x) + 1 = (x-1)^2 Q(x)$ ---- (i)

$P(x) = (x+1)^2 Q(x) + 1 \Rightarrow P(x) - 1 = (x+1)^2 Q_1(x)$ ---- (ii)

from (i), $P(x) + 1 = 0$ has $x = 1$ as 2 times repeated roots.

& $P'(x) = 0$ has 2 times repeated roots.

from (ii), $P(x) + 1 = 0$ has $x = -1$ as 2 times repeated roots.

& $P'(x) = 0$ has $x = -1$ as 2 times repeated roots.

Also, $P'(x)$ is a 4-degree poly.

∴ $P'(x) = \alpha (x-1)^2 (x+1)^2$ where 'α' is a constant

or, $P'(x) = \alpha (x^2 - 1)^2 = \alpha (x^4 - 2x^2 + 1)$

Then, $P(x) = \int P'(x)\, dx = \alpha \left[\frac{x^5}{5} - 2\frac{x^3}{3} + x\right] + c$

Now, $P(1) = -1 \Rightarrow \alpha\left(\frac{1}{5} - \frac{2}{3} + 1\right) + c = -1 \Rightarrow \frac{8\alpha}{15} + c = -1$ ---(iii)

& $P(-1) = 1 \Rightarrow \alpha\left(-\frac{1}{5} + \frac{2}{3} - 1\right) + c = 1 \Rightarrow -\frac{8\alpha}{15} + c = 1$ ---(iv) solving (iii) & (iv),

$$c = 0, \quad \alpha = -15/8$$

$$\therefore p(x) = -15/8 \left[\frac{x^5}{5} - \frac{2x^3}{3} + x\right] = -\frac{15}{8} \times \frac{x^5}{5} + \frac{15}{4} \times \frac{x^3}{3} - \frac{15}{8}x$$

i.e. $\boxed{p(x) = -3/8\, x^5 + 5/4\, x^3 - 15/8\, x}$

$\therefore$ (a) $\boxed{\text{sum of roots} = 0}$ & (b) $\boxed{\text{prod. of roots} = 0}$

Cubic polynomial :-

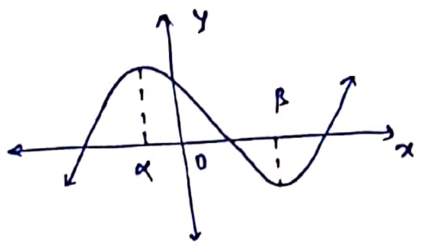
Consider, $f(x) = x^3 + ax^2 + bx + c$

$\therefore f'(x) = 3x^2 + 2ax + b$

* If $D < 0$, $f'(x) = 0$ has no real roots.
  - → $f(x)$ has no minima or maxima
  - → $f(x)$ is all time increasing (not decreasing since co-eff. of $x^3 > 0$)
  - → $f(x)$ has exactly one real root.
* when $D > 0$, $f'(x) = 0$ has real & distinct roots.
  - → $f(x)$ has a maxima & a minima.
  - → If $f'(x) = 0$ has 2 roots α, β ($\alpha < \beta$), then 'α' is local max^m & β is local min^m.

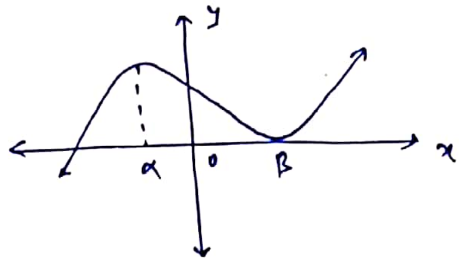
Possibilities :-

(i)



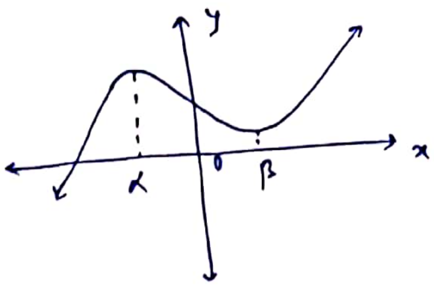
⇒ 3 distinct real roots.

(ii)



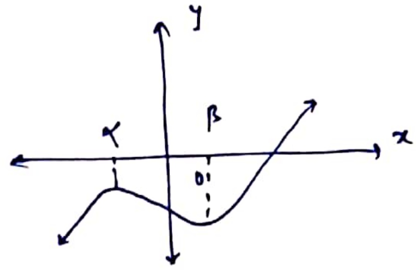
⇒ 3 roots, in which 2 roots are repeated.

(iii)



⇒ Exactly one root real

(iv)



⇒ Exactly one root real.

Students must remember:- (α, β are the roots of $f'(x) = 0$).

* $f(\alpha)\cdot f(\beta) < 0 \rightarrow$ Three distinct roots.
* $f(\alpha)\cdot f(\beta) > 0 \rightarrow$ Exactly one root
* $f(\alpha)\cdot f(\beta) = 0 \rightarrow$ 3 real roots, of that 2 coinciding

when $D = 0$, $f'(x)$ has equal roots,

so, $f'(x) = 3(x-\alpha)^2$

$\Rightarrow f(x) = \frac{3(x-\alpha)^3}{3} + c = (x-\alpha)^3 + c.$

$\therefore$ If $c = 0$, $f(x)$ has three repeated roots.

& if c is any constant then, exactly one real root is there.

26. Q. Find set of values of 'α' such that $x^3 - 12x + \alpha = 0$ has 3 distinct roots.

soln:- Let $f(x) = x^3 - 12x + \alpha \Rightarrow f'(x) = 3x^2 - 12 = 0 \Rightarrow x = \pm 2$

$\therefore f(2)\cdot f(-2) < 0 \Rightarrow (\alpha-16)(\alpha+16) < 0 \Rightarrow \boxed{\alpha \in (-16, 16)}.$

27. Q. Let a & b be the roots of $x^3 + px^2 + qx + r = 0$ with $ab+1 = 0$ then find $r^2 + pr + q = 0$.

soln:- $\therefore ab\alpha = -r$ ('α' be the another root). $\Rightarrow \alpha = r$ $(ab = -1)$

Now, $r^3 + pr^2 + qr + r = 0 \Rightarrow \boxed{r^2 + pr + q = -1}$

28. Q. If $A = \{1, 2, 3, \ldots 10\}$ & $f(x) = x^3 + ax^2 + bx + c$ where $a, b, c \in A$. then (a) find number of possible triplets (a, b, c) such that $f(x)$ is increasing.

(b) find all possible values of $a^2 + b^2 + c^2$ such that $f(x) = 0$ has 2 equal roots.

soln:- $f'(x) = 3x^2 + 2ax + b.$

For no maxima & minima, $D < 0 \Rightarrow 4a^2 - 4\times 3b < 0 \Rightarrow a^2 < 3b.$

$\therefore a = 1, b = 10$ ways.

$a = 2, b = 9$ "

$a = 3, b = 7$ "

$a = 4, b = 5$ "

$a = 5, b = 2$ "

So, possibilities of a, b is 33.

& possibilities of c is 10.

Thus no. of possible triplets $(a, b, c) = 33 \times 10 = \boxed{330.}$