FUNDAMENTALS

Begining with.....

(1.) Square Root :-

$\sqrt{1} = \pm 1$ is NOT correct.

$\sqrt{1} = 1$ is CORRECT

In fact '$\sqrt{\ }$' is the symbol for "positive square root" and '$-\sqrt{\ }$' is the symbol for negative sq. root.

$\therefore$ $\sqrt{\text{A positive no.}} \neq$ negative no.

$-\sqrt{\text{A positive no.}} \neq$ positive no.

so, $\sqrt{1} = 1$ & $-\sqrt{+1} = -1$

NOTE :-

$\sqrt{x^2} = x$ is wrong if 'x' is '-ve', because LHS $= \sqrt{x^2}$ whose value is positive while RHS $= x$ is -ve.

The right concept is

$$\Rightarrow \sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$$

↳ '$|x|$' sign gives the magnitude (absolute value) of any number.

↳ The equality sign is true for both branches, however we normally consider the first branch.

For ex:-

(i) $\sqrt{(-1)^2} = |-1| = 1$, $|-2| = 2$, $|0| = 0$, $|3| = 3$

(ii) $\sqrt{a^2 - 2a + a^2} = \sqrt{(a-1)^2} = |a-1|$

(iii) $\sqrt{\frac{1}{2}(1 + \cos 2x)} = \sqrt{\cos^2 x} = |\cos x|$

1.Q. Simplify $\sqrt{(1 - \sin\alpha \sin\beta)^2 - \cos^2\alpha \cos^2\beta}$

Soln:- Here,

$$\sqrt{(1 - \sin\alpha \sin\beta)^2 - \cos^2\alpha \cos^2\beta} = \sqrt{\sin^2\alpha - 2\sin\alpha \sin\beta + \sin^2\beta}$$

$$= \sqrt{(\sin\alpha - \sin\beta)^2}$$

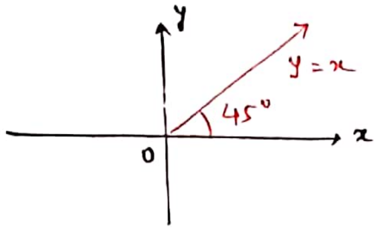
$$= |\sin\alpha - \sin\beta|$$

(2.) Modulus function :-

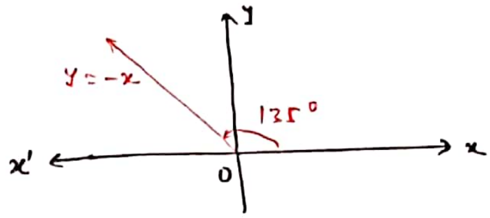
Defined as, $y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Drawing graph,

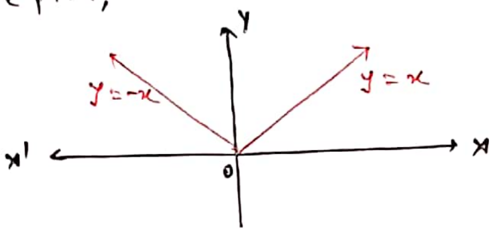
(i) $y = x,\ x \geq 0$



(ii) $y = -x,\ x < 0$



combining both the part,



Note :-

↳ As $\sqrt{x} \geq 0$, $|x|$ is also non-negative i.e. $|x| \geq 0\ \forall\, x \in R$.

↳ one should Remember that $|x|^2 = x^2\ \forall\, x \in R$.

Q.2. when $|x| > 0$?

Ans :- $x \in R - \{0\}$ i.e. for all x, except '0'.

Q.3. when $|x-1| > 0$?

Ans :- $x \in R - \{1\}$ i.e. for all x, except '1'.

Expanding modulus functions :-

ex :- $|2x-1| = \begin{cases} (2x-1), & 2x-1 \geq 0 \text{ i.e. } x \geq 1/2 \\ -(2x-1), & 2x-1 < 0 \text{ i.e. } x < 1/2 \end{cases}$

understanding Modulus by more better way using following six statements :-

(i) $|x| = 2$

(ii) $|x| = -2$

(iii) $|x| < 2$

(iv) $|x| < -2$

(v) $|x| > 2$

(vi) $|x| > -2$

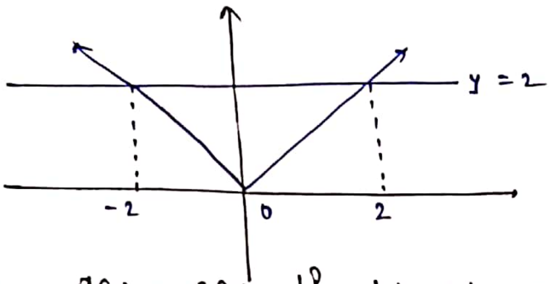
To understand these statements, we have Two ways Geometrically & GRAPHICALLY. 'Graphical method' is most efficient method.

Geometrical Meaning :-

(i) $|x| = 2$ can also be written as $|x-0| = 2$, which means that, "x is a no. at a distance 2 from 0 which certainly 2 & -2".

Similarly, we can think of rest statements.

Now,



Note :- If we have given $f(x) = g(x)$, then to solve graphically, we will draw $y = f(x)$ & $y = g(x)$ separately & then we find those value of x, for which both satisfies. i.e. intersection point of both the graphs.

$\therefore \quad |x| = \pm 2$

(ii) $|x| = -2 \quad \Rightarrow \quad$ No solution

(iii) $|x| < 2 \quad \Rightarrow \quad -2 < x < 2$

(iv) $|x| < -2 \quad \Rightarrow \quad \cancel{x > 2 \text{ or } x < -2}$ No solution

(v) $|x| > 2 \quad \Rightarrow \quad x < -2$ or, $x > 2$

(vi) $|x| > -2 \quad \Rightarrow \quad x \in R$

Generalised Result :-

$$\text{If } k > 0 \Rightarrow \begin{cases} |f(x)| = k \Rightarrow f(x) = \pm k \\ |f(x)| < k \Rightarrow -k < f(x) < k \\ |f(x)| > k \Rightarrow f(x) < -k \text{ or, } f(x) > k \end{cases}$$

$$\text{If } k < 0 \Rightarrow \begin{cases} |f(x)| = k \Rightarrow \text{No solution.} \\ |f(x)| < k \Rightarrow \text{No solution} \\ |f(x)| > k \Rightarrow x \in R. \end{cases}$$

NOTE :-

writing, $|f(x)| < k \Rightarrow f(x) < \pm k$ is WRONG and

$|f(x)| > k \Rightarrow f(x) > \pm k$ is WRONG.

(3.) $y = x^2 \quad (x^2 \geq 0)$.

It is a non-negative term. It also possess solutions in the manner of modulus as shown below.

(i) $x^2 = 4 \Rightarrow x = \pm 2$

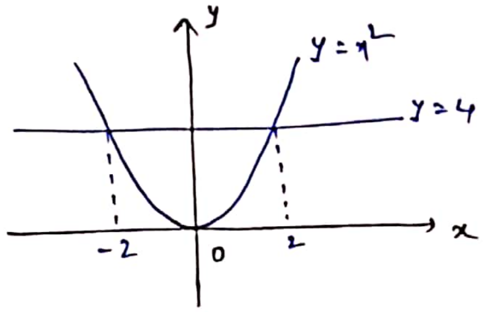
(ii) $x^2 = -4 \Rightarrow$ No solution

(iii) $x^2 < 4 \Rightarrow -2 < x < 2$

(iv) $x^2 < -4 \Rightarrow$ No solution.

(v) $x^2 > 4 \Rightarrow x < -2$ or, $x > 2$

(vi) $x^2 > -4 \Rightarrow x \in R$



Solving Modulus Inequality :-

Q.4. Solve :

(i) $|4x - 1/2| > 7$ (ii) $||x-1| + 2| = 1$ (iii) $\left|\frac{2}{x-13}\right| > 8/9$ (iv) $2 \le (2-3x)^2 \le 3$

Sol^n:-

(i) $|4x - 1/2| > 7 \Rightarrow (4x - 1/2) < -7$ or, $(4x - 1/2) > 7$

$\Rightarrow 4x < -7 + 1/2$ or $4x > 7 + 1/2$

$\Rightarrow x < \frac{-13}{8}$ or, $x > \frac{15}{8}$

so, $x \in (-\infty, -13/8) \cup (15/8, \infty)$

(ii) $||x-1| + 2| = 1 \Rightarrow |x-1| + 2 = \pm 1$

so, $|x-1| = -1$ or $|x-1| = -3$

so, No solution.

(iii) $\left|\frac{2}{x-13}\right| > 8/9 \Rightarrow \left|\frac{x-13}{2}\right| < 9/8 \Rightarrow \frac{|x-12|}{2} < 9/8 \Rightarrow |x-12| < 9/4$

$\Rightarrow -9/4 < x - 13 < 9/4$

$\Rightarrow -9/4 + 13 < x < 9/4 + 13$

$\Rightarrow -\frac{43}{4} < x < \frac{61}{4}$

$\Rightarrow x \in (-43/4, 61/4)$.

(iv) $2 \le (2-3x)^2 \le 3 \Rightarrow 2 \le (2-3x)^2$ and $(2-3x)^2 \le 3$

$\Rightarrow (2-3x)^2 \geq 2$

$\Rightarrow 2 - 3x \le -\sqrt{2}$ or, $2 - 3x \geq \sqrt{2}$

$\Rightarrow -3x \le -\sqrt{2} - 2$ or, $-3x \geq \sqrt{2} - 2$

$\Rightarrow x \geq \frac{\sqrt{2}+2}{3}$ or, $x \geq \frac{2-\sqrt{2}}{3}$ ---- (A)

$(2-3x)^2 \le 3$

$\Rightarrow -\sqrt{3} \le 2 - 3x \le \sqrt{3}$

$\Rightarrow -\sqrt{3} - 2 \le -3x \le \sqrt{3} - 2$

$\Rightarrow \frac{\sqrt{3}+2}{3} \geq x \geq \frac{2-\sqrt{3}}{3}$ --- (B) finding $A \cap B \Rightarrow x \in \left[\frac{2-\sqrt{2}}{3}, \frac{2-\sqrt{2}}{3}\right] \cup \left[\frac{2+\sqrt{2}}{3}, \frac{2+\sqrt{2}}{2}\right]$

EXERCISE :-

(i) when $|x| > x$?

(ii) when $|x| < x$?

(iii) when $|x| = -x$?

(iv) find x, if $|2x-3| = 2x-3$

(v) find x, if $|2x-5| = 5-2x$

(vi) $\frac{1}{|2x+5|} < 1/3$

(vii) $(1-2x)^2 > 4$.

PROPERTIES OF MODULUS :-

(i) $|a+b| \le |a| + |b| \longrightarrow$
- $|a+b| = |a|+|b|$, iff $a \cdot b \geq 0$
- $|a+b| < |a|+|b|$, iff $a \cdot b < 0$

(ii) ~~Hath~~ $||a|-|b|| \le |a-b| \longrightarrow$
- $|a-b| = ||a|-|b|| \Rightarrow a \cdot b \geq 0$
- $|a-b| > ||a|-|b|| \Rightarrow a \cdot b < 0$.

proof (i): $|a+b| = |a| + |b|$

$\Rightarrow (a+b)^2 = (|a|+|b|)^2$

$\Rightarrow \cancel{a^2} + \cancel{b^2} + 2ab = \cancel{a^2} + \cancel{b^2} + 2|a|\cdot|b|$

$\Rightarrow a \cdot b = |a| \cdot |b|$

$\Rightarrow$ a & b must have same sign i.e. $a \cdot b \geq 0$

similarly, you can prove for rest.

(iii) $|a \cdot b| = |a| \cdot |b|$ (iv) $\left|a/b\right| = \frac{|a|}{|b|}$ $(|b| \neq 0)$ (v) $|a^2| = |a|^2 = a^2$

(vi) $|a^n| = |a|^n$.

Remarks :- $|f(x) + g(x)| \le |f(x)| + |g(x)| \longrightarrow$
- $|f(x)+g(x)| = |f(x)| + |g(x)|$ if $f(x) \cdot g(x) \geq 0$
- $|f(x)+g(x)| < |f(x)| + |g(x)|$ if $f(x) \cdot g(x) < 0$.

Example :-

Q.5. solve: $|x-2| + |x-6| = 4$

sol^n :- we can write it as $|x-2| + |6-x| = |(x-2) + (6-x)|$

so, $(x-2)(6-x) \geq 0$

$\therefore x \in [2, 6]$

Q.6. solve : $|x| + |x-3| = 3$

sol^n :- we can write it as $|x| + |3-x| = |x + (3-x)|$

so, $x(3-x) \geq 0$

$\therefore x \in [0, 3]$

Q.7. $|x|^2 - 5|x| + 4 = 0$

Sol^n:-

Here, $|x|^2 - 5|x| + 4 = 0$

or $|x|^2 - 4|x| - |x| + 4 = 0$

or $(|x| - 4)(|x| - 1) = 0$

$\therefore$ Either $|x| = 4$ or $|x| = 1$

So, $x = \pm 1, \pm 4$.

* Students must solve : $|x|^2 + 4|x| + 3 = 0$

JEE

Q.8. Find the sum of all real roots of $|x-2|^2 + |x-2| - 2 = 0$

Sol^n:- Here, Let $|x-2| = t$

$\therefore\ t^2 + t - 2 = 0 \Rightarrow t^2 + 2t - t - 2 = 0$

$\Rightarrow (t+2)(t-1) = 0$

Either $t = -2$ or, $t = 1$

i.e. $|x-2| = -2$ or, $|x-2| = 1 \Rightarrow x - 2 = \pm 1 \Rightarrow x = 3, 1$

(No sol^n.)

So, sum $= 3 + 1 = 4$.

* Students must solve : $|x| + \left|\frac{x}{x-1}\right| = \frac{x^2}{|x-1|}$

$\left(\text{Hint : } x \cdot \frac{x}{x-1} \geq 0\right)$.

Q.9. $|x-2| - |x-6| = 10$

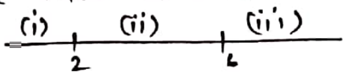
As, we can see, here no any property directly can be used. so, in this case we have two methods to solve, that are ~~Any~~ Analytical & Graphical. Again, Graphical method is used mostly, as it saves time.

Analytical sol^n:-

Step1:- Equate each modulus exp^n with zero & find critical points:-

$\therefore x = 2, 6$

Step 2:- Divide into cases:-



(3 cases)

(i) $x \leq 2$

so, $x - 2 \leq 0$ & $x - 6 \leq -4 \Rightarrow -(x-2) + (x-6) = 10 \Rightarrow -\cancel{x} + 2 + \cancel{x} - 6 = 10$ (false)

(ii) $2 < x < 6$

so, $x - 2 > 0$ & $x - 6 < 0 \Rightarrow x - 2 + x - 6 = 10 \Rightarrow 2x = 18 \Rightarrow x = 9$ (Doesn't lie in case, so no value of x).

(iii) $x \geq 6 \Rightarrow x - 2 \geq 4,\ x - 6 \geq 0 \Rightarrow \cancel{x} - 2 - \cancel{x} + 6 = 10$ (false)

So, from all three cases, there are no any values of 'x'. so, no solution.